HW 7 - Book-keeping Ex 3

- Problem Type :- (c) $E|Z_J| = \infty$. (b) $\mathbb{P}\left(Z_n = \frac{(-2)^n}{n} \,\middle|\, J > n\right) = 1$

$\{Z_n\}_{n \geq 1}$ - Martingale $\quad J$ - stopping time

$E[Z_J] = E[Z_1] \iff \underbrace{E|Z_J| < \infty \;\&\; \lim_{n \to \infty} E[Z_n | J > n]\, P(J > n) = 0}_{(OST)}$

Solution : HW7 Bk-3 :-

$X_i \equiv$ iid $X \qquad \mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$

$Z_{n+1} = Z_n \left(1 + \frac{X_n(3n+1)}{n+1}\right) \quad \forall n \geq 1, \quad Z_1 = -2$

$J = \min\{n \geq 1 \mid \text{sign}(Z_n) = \text{Sign}(Z_{n-1})\}$

(a) $\{Z_n\}_{n \geq 1}$ is a martingale

$$|Z_{n+1}| \leq |Z_n| \left(1 + |X_n| \frac{3n+1}{n+1}\right) \leq 4|Z_n| \quad \forall n \geq 1$$

- Inductively :- $|Z_{n+1}| \leq 4^n |Z_1| < \infty \quad \forall n \geq 1$

$\Rightarrow E|Z_n| < \infty \quad \forall n \geq 1$

- $E[Z_n | Z_{n-1} .. Z_1] = E[Z_n | X_{n-1} .. X_1]$

$= E\left[Z_{n-1}\left(1 + \frac{X_n(3n+1)}{n+1}\right) \,\middle|\, X_{n-1} .. X_1\right]$

$= Z_{n-1} \; E\left(1 + \frac{X_n(3n+1)}{n+1}\right) = Z_{n-1}$

$\left(\begin{array}{c} E[X] = 0 \\ X_n \overset{d}{=} X \end{array}\right)$

(b)

- $\mathbb{P}\left(Z_n = \frac{(-2)^n}{n} \,\middle|\, J > n\right) = 1$

- $\mathbb{P}\left(Z_n = \frac{-2^n(n-2)}{n(n-1)} \,\middle|\, J = n\right) = 1$

J - stopping time

$J = n \iff \text{Sign}(Z_n) = \text{Sign}(Z_{n-1})$

$\& \text{ Sig}(Z_i) \neq \text{Sign}(Z_{i-1})$

$2 \leq i \leq n-1$

$\text{Sig}(Z_2) \neq \text{Sig}(Z_1)$

$\iff X_{n-1} = 1, \; X_i = -1 \quad \forall i \leq n-2$

$Z_{n+1} = Z_n \left(1 + \frac{X_n(3n+1)}{n+1}\right) \quad \forall n \geq 1, \quad Z_1 = -2$

$J = \min\{n \geq 1 \mid \text{sign}(Z_n) = \text{Sign}(Z_{n-1})\}$

$J>n$: $\quad n=2 \quad Z_2 = Z_1\left(1+(-1)\frac{(3(1)+1)}{1+1}\right)$

$$= (-2)\left(1-\frac{4}{2}\right) = 2$$

Assume $\quad n=k, \; J>n \quad Z_k = \frac{(-2)^k}{k}$

$n=k+1, \; J>n \quad \Rightarrow \quad Z_k = \frac{(-2)^k}{k} \;\&\; X_{k+1}=-1$

$$\Rightarrow \quad Z_{k+1} = \frac{(-2)^k}{k}\left(1+(-1)\frac{3k+1}{k+1}\right)$$

$$Z_{k+1} = \frac{(-2)^{k+1}}{k+1}$$

$$\Rightarrow \quad \mathbb{P}\left(Z_k = \frac{(-2)^k}{k} \;\middle|\; J>k\right) = 1 \qquad \forall k\geq 1 \quad \square$$

$J=n \iff X_i=-1 \quad 1\leq i\leq n-2, \quad X_{n-1}=1$

$Z_{n-1} = \frac{(-2)^{n-1}}{n-1}$ (by previous result as $J>n-1$)

$$Z_n = Z_{n-1}\left(1+\frac{3(n-1)+1}{n-1+1}\right) = \frac{(-2)^n(2n-1)}{n(n-1)}$$

$$\therefore \quad \mathbb{P}\left(Z_n = \frac{(-2)^n(2n-1)}{n(n-1)} \;\middle|\; J=n\right) = 1 \quad \forall n\geq 1 \quad \square$$

(c) $J=n \iff X_i=-1 \quad 1\leq i\leq n-2, \quad X_{n-1}=1$

$$\Rightarrow \quad \mathbb{P}(J=n) = \frac{1}{2^{n-1}} \quad \forall n\geq 1 \qquad \left(P(J>n) = \frac{1}{2^{n-1}}\right)$$

$$E|Z_J| \doteq \sum_{n=1}^{\infty} E[|Z_J| \mid J=n]\, P(J=n) \doteq \sum_{n=1}^{\infty}\left|\frac{(-2)^n(2n-1)}{n(n-1)}\right|\frac{1}{2^{n-1}}$$

Ex:- Formal calculation; Justify with partial sums & limits)

$$\doteq \sum_{n=1}^{\infty}\frac{2(2n-1)}{n(n-1)} \geq \sum_{n=1}^{\infty}\frac{4}{n} = \infty$$

$$E[Z_n \mid J>n]\, P(J>n) = \frac{(-2)^n}{n}\cdot\frac{1}{2^{n-1}} = \frac{2}{n} \to 0 \text{ as } n\to\infty$$

— Martingale and stopping rule — $E|Z_J| = \infty$

Recall: Martingale Convergence Theorem-1.

$\{Z_n\}_{n\geq 1}$ - be a martingale. $\sup_{n\geq 1} E[Z_n^2] < \infty$

Then $\exists$ r.v. Z $(\equiv Z_\infty)$ such that $\mathbb{P}(Z_n \to Z \text{ as } n\to\infty) = 1$

$P(Z<\infty) = 1$

Proof: involved - Kolmogorov's maximal inequality

(Z_n^2) - sub martingale - "$Y_n = Z_{n+k} - Z_k$" $\square$

- $\{Z_n\}_{n\geq 1}$ - Cauchy w.p. 1

Theorem 3.16 (Martingale Convergence Theorem - 2)

Suppose $\{Z_n\}_{n\geq 1}$ is a martingale such that

$\sup_{n\geq 1} E|Z_n| < \infty$ then

$\exists$ a r.v. Z $(\equiv Z_\infty)$ such that

$\mathbb{P}(Z<\infty) = 1$ & $\mathbb{P}(\lim_{n\to\infty} Z_n = Z) = 1$

We have the following immediate corollary to the above result. Namely,

Corollary 3.17 (Martingale Convergence Theorem - 3)

Suppose $\{Z_n\}_{n\geq 1}$ is a non-negative martingale

$\exists$ a r.v. Z such that

$\mathbb{P}(Z<\infty) = 1$ & $\mathbb{P}(\lim_{n\to\infty} Z_n = Z) = 1$

Proof:- $E|Z_n| = E[Z_n] = E[Z_1]$

$\therefore \sup_{n\geq 1} E|Z_n| < \infty$

Apply Theorem 3.16 $\square$

Proof of Theorem 3.16: (Doob's Upcrossing Inequality)

Formal Idea:- $\{x_n\}_{n\geq 1}$ is a sequence of real numbers

$x_n \to x$ as $n \to \infty$; $\{x_n\}_{n\geq 1}$ does not converge in $\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$

$x_n \to \infty$ as $n \to \infty$

$x_n \to -\infty$ as $n \to \infty$

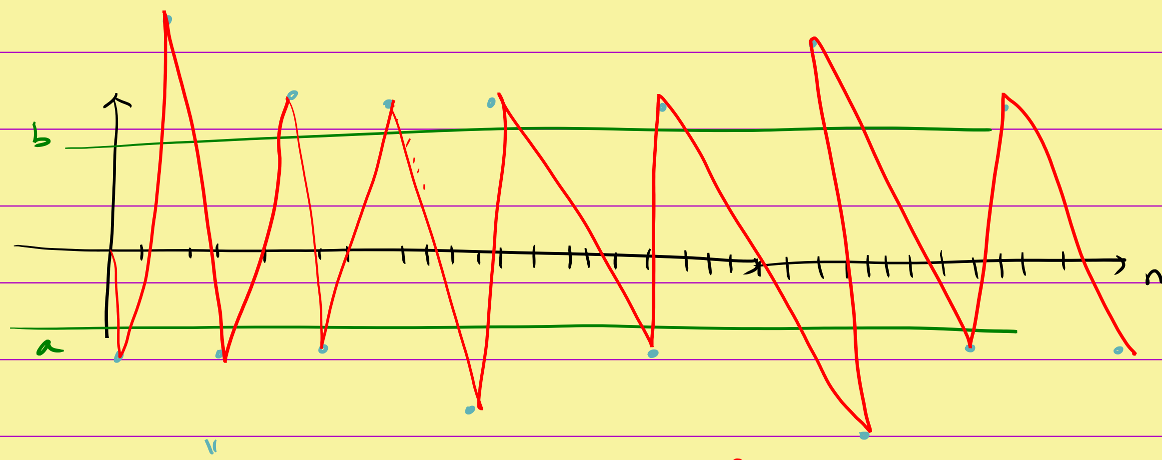
$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$

$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n$

$\exists\ a,b \in \mathbb{Q}$

$\liminf_{n\to\infty} x_n < a < b < \limsup_{n\to\infty} x_n$

$\{x_n\}_{n\geq 1}$ does not converge in $\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$ $\therefore \bigcup_{a,b\in\mathbb{Q}} \{\liminf_{n\to\infty} x_n < a < b < \limsup_{n\to\infty} x_n\}$



$\therefore$ "# crossings of (a,b) for some $a<b$, $a\in\mathbb{Q}$ $b\in\mathbb{Q}$" $= \infty$ for the sequence $\{x_n\}_{n\geq 1}$

Proof:- let $\{Z_n\}_{n\geq 1}$ be a martingale, $\sup_n \mathbb{E}|Z_n| < \infty$

For $a,b \in \mathbb{Q}$; $\Lambda_{a,b} = \{\omega\in\Omega \mid \liminf_{n\to\infty} Z_n < a < b < \limsup_{n\to\infty} Z_n\}$

$\mathbb{P}(\{Z_n\}_{n\geq 1}$ does not converge to $a \in \mathbb{R}\cup\{\infty\}\cup\{-\infty\})$

(union bound; $\{a,b\in\mathbb{Q} \mid a<b\}$ is countable)

$$= \mathbb{P}\Big(\bigcup_{\substack{a,b\in\mathbb{Q}\\ a<b}} \Lambda_{a,b}\Big) \leq \sum_{\substack{a,b\in\mathbb{Q}\\ a<b}} \mathbb{P}(\Lambda_{a,b}) \quad - (*)$$

If we show $\mathbb{P}(\Lambda_{a,b}) = 0 \quad \forall\, a,b \in \mathbb{Q} \quad a<b$ — (+)

then (*) $\Rightarrow$

$\mathbb{P}(\{Z_n\}_{n\geq 1}$ does not converge to $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}) = 0$

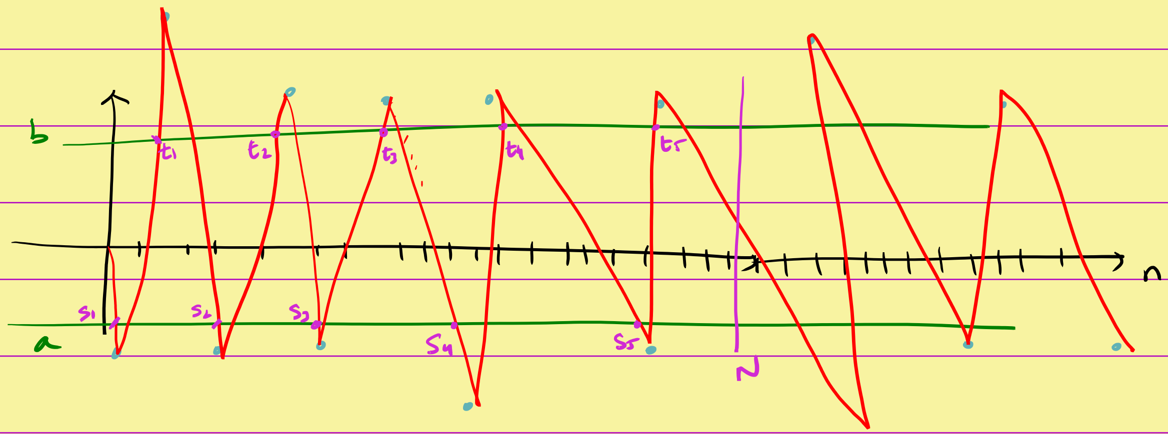
$\therefore \quad \mathbb{P}(Z_n \to Z \quad Z \in \mathbb{R} \cup \{\pm\infty\}) = 1$ — (**)

Now to show (+) let $a, b \in \mathbb{Q}$

$$\Lambda_{a,b} = \{\omega \in \Omega \mid \liminf_{n\to\infty} Z_n < a < b < \limsup_{n\to\infty} Z_n\}$$

$N \geq 1$, let $s_1 = \min\{n \geq 1 \mid Z_n \leq a\}$; $t_k = \min\{n \geq s_k \mid Z_n > b\}$

$s_k = \min\{n \geq t_{k-1} \mid Z_n \leq a\}$



$U_N(a,b)$ = # of crossings of (a,b) until time N

$= \max\{k \geq 1 \mid 1 \leq s_1 < t_1 \cdots s_k < t_k \leq N\}$

Observation:

- As $n \to \infty$ $\quad U_N(a,b) \uparrow U(a,b) \in \mathbb{R} \cup \{\infty\}$

where $U(a,b)$ = # of crossings of (a,b) by $\{Z_n\}_{n\geq 1}$.

- $\Lambda_{a,b} = \{U(a,b) = \infty\}$

$\therefore$ To show (+) it is enough to show

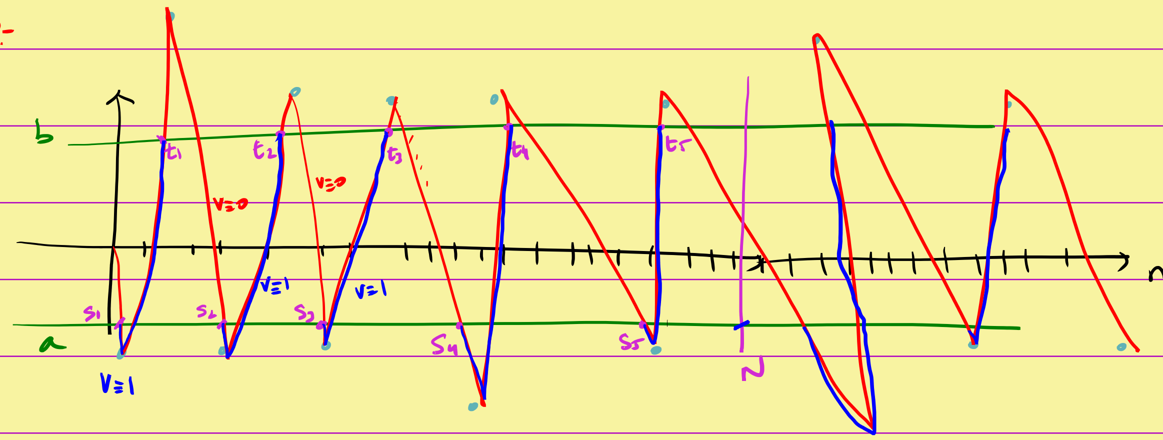
$\mathbb{P}(U(a,b) = \infty) = 0$ — (+)'

HW5 #2 : $\{Z_n\}_{n\geq 1}$ – martingale, $\{V_k = c\} \in \mathcal{A}_{k-1}$ $\Big\} \Rightarrow \sum_{k=1}^{n} V_k (Z_k - Z_{k-1})$ → Martingale. $Z_0 = 0$.

$V_1 = 1(Z_1 < a)$

$V_n = 1(V_{n-1} = 1,\ Z_{n-1} \leq b) + 1(V_{n-1} = 0,\ Z_{n-1} < a)$

Picture :-



$V_n \equiv$ 1 if below a and till you reach b

0 if above b and till you reach a

$N \geq 1$ $\quad Z_N^V = \sum_{k=1}^{N} V_k (Z_k - Z_{k-1}) \qquad Z_0 = 0$

– is a martingale. $(\{V_n = c\} \in \mathcal{A}_{n-1})$

$\left(E|Z_N^V| < \infty \Leftarrow \sum_{k=1}^{N} E|V_k|\,|Z_k - Z_{k-1}| \leq \sum_{k=1}^{N} E|Z_k| + E|Z_{k-1}| < \infty \right)$

$Z_N^V \geq (b-a)\, U_N(a,b) - (Z_N - a)^-$

$\therefore (b-a)\, E(U_N(a,b)) \leq E[Z_N^V] + E(Z_N - a)^-$

$\therefore \qquad \leq E[Z_1^V] + E[Z_1] + |a|$

$= c \qquad$ – (#)

Monotone convergence theorem :

If $0 \leq U_N(a,b) \uparrow U(a,b)$ then $E\, U_N(a,b)$

$E\, U_N(a,b) \uparrow E\, U(a,b)$

[Assume without proof]

From #

$$(b-a)\ E(U(a,b)) \leq C$$

$$\Rightarrow E(U(a,b)) < \infty$$

$$\Rightarrow \mathbb{P}(U(a,b) = \infty) = 0$$

Thus proving +

$\therefore$ it implies (**)

$$\mathbb{P}(Z_n \to Z,\ Z \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}) = 1$$

Fatou's lemma :

$$E|Z| = E[\liminf_{n\to\infty} |Z_n|]$$

(Assumption)

$$\leq \liminf_{n\to\infty} E|Z_n| \leq \sup_{n\geq 1} E|Z_n| < \infty$$

$$\Rightarrow P(|Z| = \infty) = 0 \Rightarrow \mathbb{P}(Z < \infty) = 1$$

$\square$

Revisit Theorem 3.12 [Wald's Identity] – [Tilting of Probabilities]
(Cramer's transform)

Theorem 3.12 : $\{X_i\}_{i\geq 1}$ iid X such that

for some $\delta > 0$, $E[e^{rX}] < \infty$ $\forall r \in (-\delta, \delta)$ — +

let $S_n = X_1 + X_2 + \cdots + X_n$; $\alpha > 0$, $\beta < 0$

$$J = \min\{n \geq 1 \mid S_n \geq \alpha \text{ or } S_n \leq \beta\}$$

$\forall\ \alpha \in (-\delta, \delta)$

$$E[e^{rS_J - Jg(r)}] = 1.$$

Proof: { Recall :- Verify (OST) $\longrightarrow$ Range $(X) < \infty$

$$g(r) = \ln E[e^{rX}] \quad \forall\ r \in (-\delta, \delta)$$

Assume : (+) :- X - is discrete random variable (otherwise one can use $F_X(\cdot)$ distribution and work with $dF_X(\cdot)$)

let $\mathbb{P}(X=x) \equiv$ p.m.f of X

Construct a new Probability $\mathbb{Q}$: Fix $r \in (-\delta, \delta)$

$$\mathbb{Q}(X=x) = \mathbb{P}(X=x)\, \underbrace{e^{-rx-g(r)}}_{\text{tilt}}$$

Ex:- $\mathbb{Q}$ is indeed a probability

(i.e verify: $\mathbb{Q}(\Omega)=1$, $\mathbb{Q}(A)$ - event A, $\mathbb{Q}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{Q}(A_i)$)
$A_i \cap A_j = \phi$.

Define $\{X_i\}_{i\geq 1}$ i.i.d under $\mathbb{Q}$

$$\mathbb{Q}(X_1=x_1, \ldots, X_n=x_n) := \prod_{i=1}^{n} P(X_i=x_i)\, e^{-rx_i-g(r)}$$

$$\overset{(\text{definition})}{=} \mathbb{P}(X_1=x_1, \ldots, X_n=x_n)\, e^{-r\sum_{i=1}^{n} x_i - g(r)}$$

$\{S_n\}_{n\geq 1}$ is a random walk under $\mathbb{Q}$. Further, every event in the old walk (i.e $\{S_n\}_{n\geq 1}$ under $\mathbb{P}$) exists in the new walk (i.e $\{S_n\}_{n\geq 1}$ under $\mathbb{Q}$) but with different probabilities. - Compare with previous H.W #4

$$E\left[e^{-rS_J - Jg(r)}\right] \overset{(\text{Ex.})}{=} \sum_{n=1}^{\infty} E\left[e^{-rS_J - Jg(r)} ; J=n\right] \quad \text{- non-negative series}$$

$$= \sum_{n=1}^{\infty} E\left[e^{-rS_n - ng(r)} ; J=n\right] \rightarrow$$

Formal calculation

Take partial sum and limits to do it precisely

$$= \sum_{n=1}^{\infty} E\left[e^{-\gamma S_n - n g(\cdot)} \,\middle|\, J=n\right] \mathbb{P}(J=n)$$

— understanding

$$= \sum_{n=1}^{\infty} \sum_{\{x_1,\dots,x_n\} \in \{J=n\}} \mathbb{P}(X_1=x_1,\dots,X_n=x_n)\, e^{-\gamma s_n - n g(\cdot)}$$

$$= \sum_{n=1}^{\infty} \sum_{\{x_1,\dots,x_n\} \in \{J=n\}} \prod_{i=1}^{n} \mathbb{P}(X_i=x_i)\, e^{-\gamma s_n - n g(\cdot)}$$

$$= \sum_{n=1}^{\infty} \sum_{\{x_1,\dots,x_n\} \in J=n} \mathbb{Q}(X_1=x_1,\dots,X_n=x_n)$$

$$= \sum_{n=1}^{\infty} \mathbb{Q}(J=n) = 1$$

$\mathbb{Q}$ is a Probability

Range$(J) \in \mathbb{N}$.

Solution :- 3 HW 8

X, $Y^y_{(\cdot)}$, $N_y \stackrel{d}{=} \text{Poisson}(\lambda)$, $y \in \mathbb{Z}^d$

$\xi(n,x) = \sum_{\substack{1 \le j \le N_y \\ y \in \mathbb{Z}^d}} \sum_{i=0}^{n} \mathbb{1}(Y_j^y(i) = x)$

$$\sigma^X(n) = E^{\xi}\left[(1-a)^{\sum_{i=0}^{n} \xi(i, X(i))}\right]$$

$$= \exp\left(-\gamma \sum_{y \in \mathbb{Z}^d} \omega^{a,X}(n,y)\right)$$

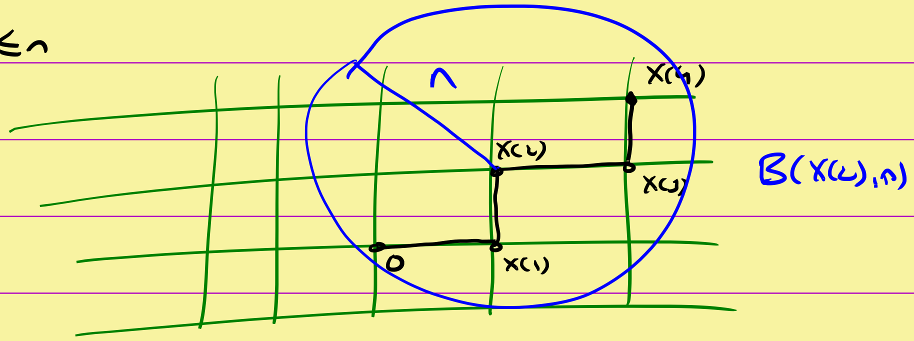
$$\omega^{a,X}(n,y) = 1 - E^Y_y\left((1-a)^{\sum_{i=0}^{n} \mathbb{1}(Y(i) = X(i))}\right)$$

Proof in Nitya's Notes : involved a formal calculation.

Assume: $\{Y_j^y(\cdot) : 1 \le j \le N_y,\ y \in \mathbb{Z}^d\}$ i.i.d

$Y \equiv$ simple symmetric R.W on $\mathbb{Z}^d$.

Given $X(i) \,:\, 0 \le i \le n$



$$K_n^d = \{ y \in \mathbb{Z}^d \mid |y - X(i)| \le n \quad i \in \{0, \dots, n\} \}$$

$$= \bigcup_{i=0}^{n} B(X(i), n)$$

$$|K_n^d| < \infty \;;\; y \in (K_n^d)^c \implies Y_j^y(i) \neq X(i) \quad \forall\, i = 0, \dots, n,\ 1 \le j \le N_y$$

$$\therefore \sum_{\substack{y \in \mathbb{Z}^d \\ 1 \le j \le N_y}} \sum_{i=0}^{n} 1(X(i) = Y_j^y(i)) = \sum_{\substack{y \in K_n^d \\ 1 \le j \le N_y}} \sum_{i=0}^{n} 1(X(i) = Y_j^y(i)) \quad \text{— } (*)$$

$$\sigma^X(n) = E^\xi\left[(1-a)^{\sum_{i=0}^{n} \xi(i, X(i))}\right]$$

$$= E^\xi\left[(1-a)^{\sum_{\substack{y \in \mathbb{Z}^d \\ 1 \le j \le N_y}} \sum_{i=0}^{n} 1(X(i) = Y_j^y(i))}\right]$$

$$= E^\xi \,(1-a)^{\sum_{\substack{y \in K_n^d \\ 1 \le j \le N_y}} \sum_{i=0}^{n} 1(X(i) = Y_j^y(i))}$$

$$= E^N E^Y \prod_{y \in K_n^d} \prod_{j=1}^{N_y} (1-a)^{\sum_{i=0}^{n} 1(X(i) = Y_j^y(i))}$$

independence

$$= \prod_{y \in K_n^d} E^N \prod_{j=1}^{N_y} \left(E^Y (1-a)^{\sum_{i=0}^{n} 1(X(i) = Y_j^y(i))} \right)$$

$$= \prod_{y \in K_n^d} E^{\mu}\left(E_y^Y (1-a)^{\sum_{i=0}^{\infty} \mathbb{1}(X(i) = Y(i))}\right)^{N_y}$$

$$= \prod_{y \in K_n^d} \left(\sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \left(E_y^Y (1-a)^{\sum_{i=0}^{\infty} \mathbb{1}(X(i) = Y(i))} \right)^k \right)$$

$$= \prod_{y \in K_n^d} e^{-\lambda \left(1 - E_y^Y\left((1-a)^{\sum_{i=0}^{\infty} \mathbb{1}(X(i) = Y(i))}\right)\right)}$$

$$\overset{*}{=} \prod_{y \in \mathbb{Z}^d} e^{-\lambda \left(1 - E_y^Y\left((1-a)^{\sum_{i=0}^{\infty} \mathbb{1}(X(i) = Y(i))}\right)\right)}$$

$$= \exp\left(-\gamma \sum_{y \in \mathbb{Z}^d} w^{a,X}(n,y)\right) \qquad \square$$