Expected hitting time of a pattern from a finite alphabet set

Section 1: Introduction

Consider a finite set Ω. Let $\{X_n\}_{n\geq 0}$ be a sequence of iid random variables with distribution Uniform(Ω). Then, for some $m \in \mathbb{N}$, consider a pattern $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_m)$ to be a finite sequence of states, where $\Psi_i \in \Omega$, $\forall\, 1 \leq i \leq m$.

Our main objective is to find the value of
$T = \min\{k \geq m, k \in \mathbb{N} : (X_{k-m+1}, \ldots, X_{k-1}, X_k) = \Psi\}$
Note that, T is the first hitting time of the pattern Ψ.
In these notes, we will describe two methods by which $E(T)$ can be computed. The first method uses the theory of Markov chains and one step probabilities. The second method uses Martingale theory.

Our main result is the following:

<u>Theorem 1</u>: Consider Ψ, a pattern of length $m \in \mathbb{N}$. Suppose there exist $0 < i_1 < i_2 < \ldots < i_j < m$, for $0 \leq j < m$, such that, the first i_k letters of the pattern Ψ is equal to the last i_k letters of the pattern Ψ, for $\forall\, 1 \leq k \leq j$. That is, $(\Psi_1, \Psi_2, \ldots, \Psi_{i_k}) = (\Psi_{m-i_k+1}, \Psi_{m-i_k+2}, \ldots, \Psi_m)$, $\forall\, 1 \leq k \leq j$. Then,

$$E(T) = |\Omega|^m + \sum_{k=1}^{j} |\Omega|^{i_k}$$

Note that $E(T)$ will depend on $|\Omega|$, m and i_k.

Section 2: Intuition and Examples

We shall discuss a few examples that illustrate the methods and then prove the final result.

Example 1: Let $\Omega = \{A, B, C, \dots, X, Y, Z\}$ and let X_n be iid Uniform(Ω), for $n \geq 1$. Let $\psi = (A, B, A)$.

Define $T = \min\{k \geq 0 : (X_{k+1}, X_{k+2}, X_{k+3}) = (A, B, A)\}$

We want to find $E(T)$. Consider the following partition of Ω^3:

$$C_1 = \{Y \in \Omega^3 : Y = (A, B, A)\}$$

$$C_2 = \{Y \in \Omega^3 : Y = (\alpha, A, B),\ \text{where } \alpha \in \Omega \text{ and } Y \notin C_1\}$$

$$C_3 = \{Y \in \Omega^3 : Y = (\alpha, \beta, A),\ \text{where } \alpha, \beta \in \Omega \text{ and } Y \notin C_1, C_2\}$$

$$C_4 = \{Y \in \Omega^3 : Y = (\alpha, \beta, \gamma),\ \text{where } \alpha, \beta, \gamma \in \Omega \text{ and } Y \notin C_1, C_2, C_3\}$$

Let $Y_n = (X_n, X_{n+1}, X_{n+2})$, for $n \geq 1$.

One may view $\{Y_n\}_{n \geq 0}$ as a Markov chain on $S = \{C_1, C_2, C_3, C_4\}$ with the following transition matrix.

	C_4	C_3	C_2	C_1
C_4	$25/26$	$1/26$	0	0
C_3	$24/26$	$1/26$	$1/26$	0
C_2	$25/26$	0	0	$1/26$
C_1	0	0	0	1

Where, given $Y_n = (X_n, X_{n+1}, X_{n+2}) \in C_i$, p_{ij} is the probability of $Y_{n+1} = (X_{n+1}, X_{n+2}, X_{n+3}) \in C_j$.

Define the stopping time $T^{c_1} = \min\{k \geq 0 : Y_k = c_1\}$ and for $c \in S$, $h(c) = E(T^{c_1} : Y_0 = c)$, that is $h(c)$ is the expected hitting time of the state c_1, given that the Markov chain starts from the state c.

Using one step probability calculation (see HW2 Problem 2), we have

$$h(c_4) = 1 + \frac{25}{26} h(c_4) + \frac{1}{26} h(c_3) + 0 \times h(c_2) + 0 \times h(c_1)$$

$$h(c_3) = 1 + \frac{24}{26} h(c_4) + \frac{1}{26} h(c_3) + \frac{1}{26} h(c_2) + 0 \times h(c_1)$$

$$h(c_2) = 1 + \frac{25}{26} h(c_4) + 0 \times h(c_3) + 0 \times h(c_2) + 1 \times h(c_1)$$

$$h(c_1) = 0$$

We have $h(c_4) = 1 + \frac{25}{26} h(c_4) + \frac{1}{26} h(c_3)$;

$$h(c_3) = 1 + \frac{24}{26} h(c_4) + \frac{1}{26} h(c_3) + \frac{1}{26} + \frac{25}{26^2} h(c_4)$$

$$\Rightarrow \quad \frac{1}{26} h(c_4) = 1 + \frac{1}{26} h(c_3) \text{ ;}$$

$$\frac{25}{26} h(c_3) = \frac{27}{26} + h(c_4)\left(\frac{24}{26} + \frac{25}{26^2}\right)$$

Combining the above equations we have,

$$\frac{25}{26} h(c_4) = 25 + \frac{27}{26} + h(c_4)\left(\frac{24}{26} + \frac{25}{26^2}\right)$$

$$\Rightarrow \quad h(c_4)\left(\frac{1}{26^2}\right) = \frac{25 \times 26 + 27}{26}$$

$$\Rightarrow h(C_4) = 26^2(25) + 27(26) = 26^2(26-1) + 26(26+1) = 26^3 + 26$$

$$h(C_3) = 26^3 \; ; \; h(C_2) = 26^3 - 26^2 + 26 \; ; \; h(C_1) = 0$$

We know that, $E(T^{C_1}) = \sum_{i=1}^{4} \mathbb{P}(Y_0 = C_i)\, E(T^{C_1} \mid Y_0 = C_i)$

$= \sum_{i=1}^{4} \mathbb{P}(Y_0 = C_i)\, h(C_i)$, where $\mathbb{P}(Y_0 = C_i) = \dfrac{|C_i|}{|\Omega^3|} = \dfrac{|C_i|}{26^3}$ [since X_n is uniform over Ω]

$$= h(C_1) \times \frac{|C_1|}{26^3} + h(C_2) \times \frac{|C_2|}{26^3} + h(C_3) \times \frac{|C_3|}{26^3} + h(C_4) \times \frac{|C_4|}{26^3}$$

$$= \frac{1}{26^3}\left(0(1) + (26 + 25(26)^2)(26) + 26^3(26^2-1) + (26^3+26)(26^3-26^2-26) \right)$$

$$= 26^3 + 26 - 3$$

$\therefore$ $E(T^{C_1}) = 26^3 + 26 - 3$. But it takes 3 units of time to reach the initial state s_0 of this markov chain.

$\therefore$ Expected hitting time of the pattern (A, B, A) =

$$E(T) = E(T^{C_1}) + 3 = 26^3 + 26 - 3 + 3 = 26^3 + 26 \quad \square$$

Note that $|\psi| = 3$ in the above example and we needed to solve a 4×4 system of linear equations. Then if $|\psi| = n$, we would have to solve an $(n+1) \times (n+1)$ system of linear equations, which can get very calculation intensive. So, this method may become untractable.

We now describe another example where $|\psi| = 11$ and uses martingale theory.

Example 2: Let $\Omega = \{A, B, C, \dots, X, Y, Z\}$ and
$\psi = (A, B, R, A, C, A, D, A, B, R, A)$.

Let us consider a typewriter with the 26 english alphabets on it. Once every second (k), a monkey randomly types a letter X_k, from the typewriter. This setup is inside a casino.

At every time k, $k \geq 1$, a new gambler walks into the casino and bets ₹1 that X_k, the next letter the monkey will type is going to be 'A'. If they are correct, they win ₹26 otherwise they leave the game.

Why ₹26 ? Because the bet has to be fair.

$$\text{Expected profit} = \frac{1}{26}(26-1) + \frac{25}{26}(-1) = 0$$

If the gambler was correct, then in the next time step (next round) they bet the entire ₹26 that the upcoming letter, X_{k+1} will be 'B' (for a prize of ₹26^2) and so on.

This goes on until the word ABRACADABRA appears and then the game stops.

Let $\{X_n\}_{n\geq 1}$ be iid Uniform (Ω) random variables.

Define $T = \min\{k \geq 11 : (X_{k-10}, \dots, X_k) = \psi\}$.
Clearly, T is a stopping time w.r.t the filtration $\mathcal{A}_n$ defined by $\{X_n\}_{n\geq 1}$, since the stopping time T being equal to n is completely determined by the values $\{X_1, X_2, \dots, X_n\}$.

Let Z_n^k be the amount of money the k^{th} gambler has at the end of time n.
Then, for $n \in \mathbb{N} \cup \{0\}$,

$$Z_n^k = \begin{cases} 1 & , \text{ if } n < k \\ 26\, Z_{n-1}^k \, \mathbb{1}_{\{X_n = \Psi_{n-k+1}\}} & , \text{ if } k \leq n \leq k+10 \\ Z_{k+10}^k & , \text{ if } n \geq k+11 \end{cases}$$

Then, the net profit the casino makes from the k^{th} gambler at the end of the n^{th} round is: $Z_0^k - Z_n^k = 1 - Z_n^k$

Therefore, the total profit the casino makes from all the gamblers, at the end of the n^{th} round is:

$$M_n = \sum_{k=1}^{n} (1 - Z_n^k) = n - \sum_{k=1}^{n} Z_n^k$$

In the proof of the main result we will show that $\{M_n\}_{n\geq 1}$ is indeed a martingale. We would like to apply OST to $\{M_n\}_{n\geq 1}$.

First, we shall show that $E(T) < \infty$.

We will bound T by a random variable whose expectation is easier to calculate and hence show that $E(T)$ is also finite.

Divide the sequence of random variables $\{X_n\}_{n\geq 0}$ into blocks of size 11. If a block is equal to the pattern Ψ, it is a success and a failure otherwise. The probability of a success is $\frac{1}{26^{11}}$, i.e, each block is a Bernoulli trial.

Let S be the number of blocks until the first success.

Then, $S = k \Rightarrow T \leq 11k$, because the first appearance of Ψ could start in the middle of a block and end in the middle of another.

$$\Rightarrow E(T) \leq 11\, E(S), \text{ but } S \sim \text{Geometric}\left(\frac{1}{26^{11}}\right)$$

$$\Rightarrow E(T) \leq 11(26)^{11}$$

$$\therefore E(T) < \infty \qquad — ①$$

Secondly, we show that M_n has bounded increments.

For $n>1$, consider $|M_n - M_{n-1}| = |n - \sum_{k=1}^{n} Z_n^k - n+1 + \sum_{k=1}^{n-1} Z_{n-1}^k|$

$$= \left|\sum_{k=1}^{n-1} (Z_{n-1}^k - Z_n^k) - (Z_n^n - 1)\right| \leq \left|\sum_{k=1}^{n-1} (Z_{n-1}^k - Z_n^k)\right| + |Z_n^n - 1|$$

$$\leq \left|\sum_{k=n-10}^{n-1} (Z_{n-1}^k - Z_n^k)\right| + 25 \quad [\text{Since } Z_{n-1}^k = Z_n^k, \text{ for } n \geq k+10 \text{ and } Z_n^n \leq 26]$$

$$= \left|\sum_{k=n-10}^{n-1} (Z_{n-1}^k - 26\, Z_{n-1}^k\, 1_{\{x_n = \psi_{n-k+1}\}})\right| + 25$$

$$\leq \left|\sum_{k=n-10}^{n-1} -25 Z_{n-1}^k\right| + 25 \leq 25(1 + 26 + 26^2 + \dots + 26^{10}) \quad [\text{Since } Z_n^{n-k+1} \leq 26^k]$$

$\Rightarrow |M_n - M_{n-1}| \leq 26^{11} - 1$, $\forall n \in \mathbb{N}$ and hence, M has bounded increments. — ②

We will prove the following result in the later sections:

For a Martingale M and a stopping time T, $E(T)$ being finite and M having bounded increments $\Rightarrow$ The two conditions $(*)$ and $(**)$ which imply the Optional Stopping Theorem.

where $(*)$ = $E(M_n | T>n)\, \mathbb{P}(T>n) \to 0$ as $n \to \infty$

$(**)$ = $E(|M_T|) < \infty$

[Result from class notes lecture 12]

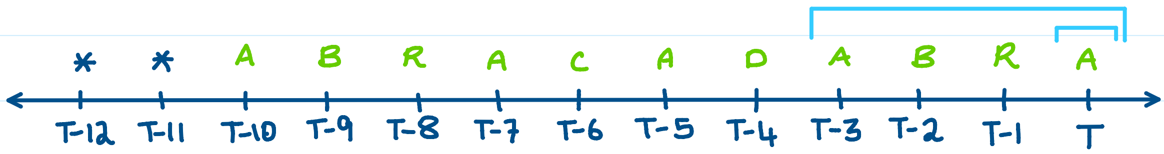
Hence, using OST, we have:

$$E(M_T) = E(M_1) = 1 - E(Z_1^1) = 1 - \left(\frac{1}{26}(26) + \frac{25}{26}(0)\right) = 0 \quad — ③$$

We now make a key observation:

$$M_T = T - \sum_{k=1}^{T} Z_T^k = T - \sum_{k=T-10}^{T} Z_T^k$$

This is indeed the case, as for all gamblers who started betting before time $T-10$, $Z_T^k = 0$.



Clearly, $Z_T^k = 0$ for $T-10 \le k \le T$ and $k \notin \{T, T-3, T-10\}$.

Then, $Z_T^T = 26$, $Z_T^{T-3} = 26^4$, $Z_T^{T-10} = 26^{11}$

Therefore, $M_T = T - 26^{11} - 26^4 - 26$

From ③, $E(M_T) = 0 = E(T - 26^{11} - 26^4 - 26)$

$$\Rightarrow E(T) = 26^{11} + 26^4 + 26$$

Section 3: Proof of Theorem 1

Now, let us consider an arbitrary alphabet set Ω of finite size and an arbitrary word Ψ of finite length.

Let us consider a typewriter with the letters from the set Ω on it. Once every second (k), a monkey randomly types a letter X_k, from the typewriter. This setup is inside a casino.

At every time k, $k \geq 1$, a new gambler walks into the casino and bets ₹1 that X_k, the next letter the monkey will type is going to be 'A'. If they are correct, they win ₹$|\Omega|$ otherwise they leave the game.

If the gambler was correct, then in the next time step (next round) they bet the entire ₹$|\Omega|$ that the upcoming letter, X_{k+1} will be 'B' (for a prize of ₹$|\Omega|^2$) and so on.

This goes on until the word Ψ appears and then the game stops.

Let $\{X_n\}_{n\geq 1}$ be iid Uniform(Ω) random variables.

Define $T = \min\{k \geq |\Psi| : (X_{k-|\Psi|+1}, \dots, X_k) = \Psi\}$
Clearly, T is a stopping time w.r.t the filtration $\mathcal{A}_n$ defined by $\{X_n\}_{n\geq 1}$, since the stopping time T being equal to n is completely determined by the values $\{X_1, X_2, \dots, X_n\}$.

Let Z_n^k be the amount of money the k^{th} gambler has at the end of time n.
Then, for $n \in \mathbb{N} \cup \{0\}$,

$$Z_n^k = \begin{cases} 1 & , \text{ if } n < k \\ |\Omega| \, Z_{n-1}^k \, \mathbb{1}_{\{X_n = \psi_{n-k+1}\}} & , \text{ if } k \le n \le k + |\psi| - 1 \\ Z_{k+|\psi|-1}^k & , \text{ if } n \ge k + |\psi| \end{cases}$$

Then, the net profit the casino makes from the k^{th} gambler at the end of the n^{th} round is: $Z_0^k - Z_n^k = 1 - Z_n^k$

Therefore, the total profit the casino makes from all the gamblers, at the end of the n^{th} round is:

$$M_n = \sum_{k=1}^{n} (1 - Z_n^k) = n - \sum_{k=1}^{n} Z_n^k$$

$$E(|M_n|) \le n + \sum_{k=1}^{n} E(|Z_n^k|) \le n + \sum_{k=1}^{n} |\Omega|^{n-k+1} < \infty \quad , \forall n \ge 1$$

Consider $E(M_n | \mathcal{A}_{n-1}) = E(M_n | X_{n-1} = x_{n-1}, \dots, X_1 = x_1) = \circledast$

$$= E\Big(n - \sum_{k=1}^{n} Z_n^k \,\Big|\, X_{n-1} = x_{n-1}, \dots, X_1 = x_1\Big)$$

$$= E\Big(\Big(n-1 - \sum_{k=1}^{n-1} Z_n^k\Big) + (1 - Z_n^n) \,\Big|\, X_{n-1} = x_{n-1}, \dots, X_1 = x_1\Big)$$

But $Z_n^k = |\Omega| \, Z_{n-1}^k \, \mathbb{1}_{\{X_n = \psi_{n-k+1}\}}$

$$\Rightarrow \circledast = E\Big(\Big(n-1 - \sum_{k=1}^{n-1} |\Omega| Z_{n-1}^k \mathbb{1}_{\{X_n = \psi_{n-k+1}\}}\Big) + (1 - Z_n^n) \,\Big|\, X_{n-1} = x_{n-1}, \dots, X_1 = x_1\Big)$$

$$= n-1 - \sum_{k=1}^{n-1} E\Big(|\Omega| Z_{n-1}^k \mathbb{1}_{\{X_n = \psi_{n-k+1}\}} \,\Big|\, X_{n-1} = x_{n-1}, \dots, X_1 = x_1\Big)$$

$$+ E\Big(1 - Z_n^n \,\Big|\, X_{n-1} = x_{n-1}, \dots, X_1 = x_1\Big)$$

But Z_{n-1}^{k} is determinable by time $n-1$, that is through the values of $\{X_1, X_2, \dots, X_{n-1}\}$

$$\Rightarrow (*) = n-1 - \sum_{k=1}^{n-1} |\Omega| Z_{n-1}^{k} E(\mathbb{1}_{\{X_n = \psi_{n-k+1}\}} / X_{n-1} = x_{n-1}, \dots, X_1 = x_1)$$

$$+ \; 1 - E(|\Omega| Z_{n-1}^{n} \mathbb{1}_{\{X_n = \psi_1\}} / X_{n-1} = x_{n-1}, \dots, X_1 = x_1)$$

$$= n-1 - \sum_{k=1}^{n-1} |\Omega| Z_{n-1}^{k} \left(\frac{1}{|\Omega|} \times 1 + \frac{|\Omega|-1}{|\Omega|} \times 0\right) + 1 - E(|\Omega| \mathbb{1}_{\{X_n = \psi_1\}} / X_{n-1} = x_{n-1}, \dots, X_1 = x_1)$$

[since $Z_{n-1}^{n} = 1$]

$$= \left(n-1 - \sum_{k=1}^{n-1} Z_{n-1}^{k}\right) + 1 - \left(|\Omega| \times \frac{1}{|\Omega|} + 0 \times \frac{|\Omega|-1}{|\Omega|}\right)$$

$$= M_{n-1} + 1 - 1 = M_{n-1}$$

$\therefore E(M_n | \mathcal{A}_{n-1}) = M_{n-1}$ and hence M_n is a martingale.

We would like to apply OST to $\{M_n\}_{n \geq 1}$.

Firstly, we shall show that $E(T) < \infty$.

We will bound T by a random variable whose expectation is easier to calculate and hence show that $E(T)$ is also finite.

Divide the sequence of random variables $\{X_n\}_{n \geq 0}$ into blocks of size $|\psi|$. If a block is equal to the pattern ψ, it is a success and a failure otherwise. The probability of a success is $\frac{1}{|\Omega|^{|\psi|}}$, that is, each block is a Bernoulli trial.

Let S be the number of blocks until the first success. Then, $S = k \Rightarrow T \leq |\psi| k$, because the first appearance of ψ could start in the middle of a block and end in the middle of another.

$$\Rightarrow E(T) \leq |\Psi| E(S), \text{ but } S \sim \text{Geometric}\left(\frac{1}{|\Omega|^{|\Psi|}}\right)$$

$$\Rightarrow E(T) \leq |\Psi| |\Omega|^{|\Psi|}$$

$$\therefore E(T) < \infty \qquad -①$$

Secondly, we show that M_n has bounded increments.

For $n>1$, consider $|M_n - M_{n-1}| = |n - \sum_{k=1}^{n} Z_n^k - n + 1 + \sum_{k=1}^{n-1} Z_{n-1}^k|$

$$= |\sum_{k=1}^{n-1}(Z_{n-1}^k - Z_n^k) - (Z_n^n - 1)| \leq |\sum_{k=1}^{n-1}(Z_{n-1}^k - Z_n^k)| + |Z_n^n - 1|$$

$$\leq |\sum_{k=n-|\Psi|+1}^{n-1}(Z_{n-1}^k - Z_n^k)| + |\Omega| - 1 \quad [\text{since } Z_{n-1}^k = Z_n^k, \text{ for } n \geq k + |\Psi| - 1 \text{ and } Z_n^n \leq |\Omega|]$$

$$= |\sum_{k=n-|\Psi|+1}^{n-1}(Z_{n-1}^k - |\Omega| Z_{n-1}^k \mathbb{1}_{\{X_n = \Psi_{n-k+1}\}})| + |\Omega| - 1$$

$$\leq |\sum_{k=n-|\Psi|+1}^{n-1} -(|\Omega|-1) Z_{n-1}^k| + |\Omega| - 1 \leq (|\Omega|-1)(1 + |\Omega| + \ldots + |\Omega|^{|\Psi|-1}) \quad [\text{Since } Z_n^{n-k+1} \leq |\Omega|^k]$$

$$\Rightarrow |M_n - M_{n-1}| \leq |\Omega|^{|\Psi|} - 1, \forall n \in \mathbb{N} \text{ and hence, } M \text{ has bounded increments.} \quad -②$$

Lemma: For a Martingale M and a stopping time T, $E(T)$ being finite and M having bounded increments $\Rightarrow$ The two conditions (*) and (**) which imply the Optional Stopping Theorem.

where (*) : $E(M_n | T > n) \mathbb{P}(T > n) \to 0$ as $n \to \infty$

(**) : $E(|M_T|) < \infty$

[Result from class notes Lecture 12]

Proof: Given, $E(T) < \infty$ and $|M_n - M_{n-1}| \le C$, $\forall n \ge 1$

Then, $|M_n| \le |M_n - M_{n-1}| + |M_{n-1}| \le C + |M_{n-1}|$

On doing this inductively, we have

$$|M_n| \le (n-1)C + |M_1|$$

But here, $|M_1| \le 26$

$$\Rightarrow |M_n| \le (n-1)C + 26$$

So, $|E(M_n \mid T > n)| P(T>n) \le E(|M_n| \mid T>n) P(T>n)$
$$\le ((n-1)C + 26) P(T>n)$$

Since $E(T) = \sum_{k=1}^{\infty} k P(T=k) = \sum_{k=0}^{\infty} P(T>k) < \infty$

$$\Rightarrow P(T>k) \to 0 \text{ as } k \to \infty$$

And using the Cauchy condensation test, we have that

$$E(T) = \sum_{k=0}^{\infty} P(T>k) \le \sum_{k=0}^{\infty} 2^k P(T>2^k) \le 2\sum_{k=0}^{\infty} P(T>k) = 2E(T)$$

Since $E(T)$ is finite, the series $\sum_{k=0}^{\infty} 2^k P(T>2^k)$ also converges to a finite value.

$$\Rightarrow \lim_{k\to\infty} 2^k P(T>2^k) = 0$$

Also, we have the following inequality:

$$\frac{1}{2}\left(2^{k+1} P(T > 2^{k+1})\right) \le k P(T>k) \le \left(2^k P(T>2^k)\right) \cdot 2$$

As $k \to \infty$, $2^k P(T>2^k) \to 0 \Rightarrow \lim_{k\to\infty} k P(T>k) = 0$

And hence, we have that $E(T) < \infty \Rightarrow \mathbb{P}(T>n) \to 0$ as $n \to \infty$
and $n\mathbb{P}(T>n) \to 0$ as $n \to \infty$

So, $$E(M_n \mid T>n)\mathbb{P}(T>n)$$
$$\leq cn\mathbb{P}(T>n) + (c+2b)\mathbb{P}(T>n)$$

Clearly,

$$0 \leq \lim_{n\to\infty} E(M_n \mid T>n)\mathbb{P}(T>n) \leq \lim_{n\to\infty} \left(cn\mathbb{P}(T>n) + (c+2b)\mathbb{P}(T>n)\right) = 0$$

$$\Rightarrow E(M_n \mid T>n)\mathbb{P}(T>n) \to 0 \text{ as } n \to \infty \quad (*)$$

Now, $$E(|M_T|) = E\left(\sum_{k=1}^{\infty} |M_k| 1_{\{T=k\}}\right) \leq E\left(\sum_{k=1}^{\infty} (c(k-1) + 2b) 1_{\{T=k\}}\right)$$

$$\leq c\sum_{k=1}^{\infty} k\,\mathbb{P}(T=k) + (c+2b)E\left(\sum_{k=1}^{\infty} 1_{\{T=k\}}\right)$$

$$\leq cE(T) + c + 2b < \infty$$

And hence $E(|M_T|) < \infty$ $(**)$

So, $(*)$ and $(**)$, the conditions for OST are met.

Hence, $E(M_T) = E(M_1) = 1 - E(Z_1^1) = 0$ — (3)

We now make a key observation:

$$M_T = T - \sum_{k=1}^{T} Z_T^k = T - \sum_{k=T-|\psi|+1}^{T} Z_T^k$$

This is indeed the case, as for all gamblers who started betting before time $T-|\psi|+1$, $Z_T^k = 0$.

Clearly, $Z_T^k = 0$ for $T-|\Psi|+1 \le k \le T$ and $k \notin \{T, T-i_1+1, \ldots, T-i_j+1\}$

Where, $0 < i_1 < i_2 \ldots < i_j < m$, for $0 \le j < m$ are natural numbers such that,
$(\Psi_1, \Psi_2, \ldots, \Psi_{i_k}) = (\Psi_{m-i_k+1}, \Psi_{m-i_k+2}, \ldots, \Psi_m)$, $\forall\ 1 \le k \le j$.

Then, for $1 \le k \le j$, $Z_T^{i_k - 1} = |\Omega|^{i_k}$ and $Z_T^{T-|\Psi|+1} = |\Omega|^{|\Psi|}$

Therefore, $M_T = T - |\Omega|^{|\Psi|} - \sum_{k=1}^{j} |\Omega|^{i_k}$

Using ③, $E(M_T) = 0 = E\left(T - |\Omega|^{|\Psi|} - \sum_{k=1}^{j} |\Omega|^{i_k}\right)$

$$\therefore\ E(T) = |\Omega|^m + \sum_{k=1}^{j} |\Omega|^{i_k} \quad \square$$

References:

1) Martingale Theory and Applications - Nic Freeman
2) The Abracadabra Problem - Francesco Caravena
3) Probability with Martingales - section 10.11 - David Williams