Simulate Random Variables with a given distribution

Key :- We have a generator that provides a sample from uniform $(0,1)$

Assume key exists. Call it U

$X \stackrel{d}{=} \text{Bernoulli}(p) \qquad 0 < p < 1$

- Generate U
- $X = \begin{cases} 1 & U < p \\ 0 & U \geq p \end{cases}$

Check :-

$\text{Range}(X) = \{0,1\}$

$P(X=1) = P(U<p)$

$= \frac{p-0}{1-0} = p$

Similarly

$P(X=0) = 1-p$

- $X \overset{d}{=} \text{Binomial}(n, p)$

Theorem: $X_i \sim \text{Bernoulli}(p)$ $1 \le i \le n$

$$X \overset{d}{=} \sum_{i=1}^{n} X_i$$

- Generate $U_1, U_2, \ldots U_n$ (independent)
- $X = \sum_{i=1}^{n} \mathbb{1}(U_i < p)$

Ex:

$X \overset{d}{=} \text{Binomial}(n, p)$

- X is a discrete random variable

$(\Omega, \mathcal{F}, \mathbb{P})$ - Probability space

Sample

$X : \Omega \to S$ $\quad S \subseteq \mathbb{R}$ $\quad S$ - countable

$$\text{Range}(X) = \{x_i\}_{i \geq 1} \qquad x_i \in S \quad \forall i \geq 1$$

$$\mathbb{P}(X = x_i) = p_i \qquad \text{with} \quad \sum_{i=1}^{\infty} p_i = 1$$

F

- Generate U

Ex $X \stackrel{d}{=} F$

- $X = \begin{cases} x_1 & \text{if } U < p_1 \\ x_k & \text{if } k > 1 \quad \sum_{i=1}^{k-1} p_i \leq U < \sum_{i=1}^{k} p_i \end{cases}$

- X is continuous random variable

$$\mathbb{P}(X = x) = 0 \quad \forall x \in \mathbb{R}$$

$$\mathbb{P}(a \leq X < b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

p.d.f

Where $F(t) = \mathbb{P}(X \leq t)$

c.d.f