

least Square Method

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

$$\epsilon_i \stackrel{d}{=} N(0, \sigma^2) \text{ - independent}$$

Same
Variance

Minimized

$$\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Method

Same
VarianceQuestions :-

- Is linear Relationship the correct one?

- Is $\beta_1 = 0$?

Hypothesis
TestWe will use $\hat{\beta}_1$ to understand the aboveObservation , $(\sum_{i=1}^n (x_i - \bar{x}) = 0)$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\begin{aligned} & \{y_1, y_2 \text{ independent}\} \\ & aY_1 + bY_2 \\ & \text{Var}(aY_1 + bY_2) \\ & = a^2 \text{Var}(Y_1) + b^2 \text{Var}(Y_2) \end{aligned}$$

$$\stackrel{d}{=} N(\underline{(1)}, \underline{(2)})$$

$$(\beta_1, x_1)^2$$

$$\bar{y} - \hat{\beta}_1 \bar{x}$$

$$Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2) \quad \text{independent}$$

$$E[Y_i] = \beta_0 + \beta_1 x_i$$

$$\text{Var}[Y_i] = \sigma^2$$

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\textcircled{1} = E[\hat{\beta}_1] = \frac{\sum_{i=1}^n (x_i - \bar{x})(\beta_0 + \beta_1 x_i)}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$= \frac{\sum_{i=1}^n (x_i - \bar{x}) \beta_0}{\sum_{i=1}^n (x_i - \bar{x})^2} + \beta_1 \frac{\sum_{i=1}^n (x_i - \bar{x}) x_i}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$= 0 + \beta_1 1 = \beta_1 \quad \left[\hat{\beta}_1 - \text{unbiased estimator for } \beta_1 \right]$$

up the

the sum
test

and the above

$$\begin{cases} x_1, x_2 \text{ indep} \\ aY_1 + bY_2 \\ \text{Var}(aY_1 + bY_2) \\ = a^2 \text{Var}(Y_1) + b^2 \text{Var}(Y_2) \end{cases}$$

$$N(\textcircled{1}, \textcircled{2})$$

What is done if σ is unknown?

Residual $y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \quad 1 \leq i \leq n$

Sum
of squares

RSS

$$= \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

Can we estimate σ^2 ?

Result 1:-

$$E[RSS] = (n-2) \sigma^2$$

Proof: - Next class $\square$

unknown?

x_i $1 \leq i \leq n$

$\hat{\beta}_1 x_i)^2$

$$\frac{\hat{\beta}_1 - \textcircled{1}}{\sqrt{\textcircled{2}}} \stackrel{d}{=} N(0,1)$$

$$\textcircled{1} = \beta_1$$

$$\textcircled{2} = \frac{\sigma^2}{S_{xx}}$$

Result 2:- RSS & $\hat{\beta}_1$
are independent

Result (3)

$$\frac{\text{RSS}}{\sigma^2} \stackrel{d}{=} \chi^2_{n-2}$$

χ^2_{n-2} - chi-square with $n-2$ degrees
of freedom

$$① = \beta$$

$$② = \frac{\sigma^2}{S_{xx}}$$

Example 8.18 $U \stackrel{d}{=} N(0,1)$ & $V \stackrel{d}{=} \chi^2_n$ Independent

Book on website $\Rightarrow \frac{U}{\sqrt{\frac{V}{n}}} \stackrel{d}{=} t_n$ (Student's t)

$$\Rightarrow \frac{\hat{\beta}_1 - ①}{\sqrt{②}} \bigg/ \sqrt{\frac{RSS}{\sigma^2(n-2)}} \stackrel{d}{=} t_{n-2}$$

$$\Rightarrow \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{RSS}{(n-2)S_{xx}}}} \stackrel{d}{=} t_{n-2}$$

Applications

- Use t_{n-2} to get confidence intervals for β_1

$$\chi^2_{n-2}$$

degrees