Question: Assume that for $\lambda > 0$, $l \in \mathbb{N}$,

$$\frac{\lambda^l}{l-1!}\int_0^1 z^{l-1}e^{-\lambda z}dz = \sum_{m=l}^{\infty} e^{-\lambda}\frac{(\lambda)^m}{m!}$$

Let $\{U_n\}_{n\geq 1}$ be Uniform $(0,1)$ random variables. Let

$$X = \begin{cases} 0 & \text{if } \frac{-1}{\lambda}\ln(U_1) > 1 \\ \max\{j \geq 1 : \frac{-1}{\lambda}\ln(U_1U_2\ldots U_j) \leq 1\} & \text{otherwise} \end{cases}$$

Find the distribution[1] of X

Answer: First note that,

$$\text{Range}\{X\} = \{0\} \cup \mathbb{N}. \tag{1}$$

Observe that

$$\mathbb{P}(X=0) = \mathbb{P}(\frac{-1}{\lambda}\ln(U_1) > 1) = \mathbb{P}(U_1 \leq e^{-\lambda}) = e^{-\lambda} \tag{2}$$

For $i \geq 1$, let $T_i = -\frac{1}{\lambda}\ln(U_i)$. Then $T_i \sim \text{Exp}(\lambda)$ and consequently

$$\frac{-1}{\lambda}\ln(U_1U_2\ldots U_j) = \frac{-1}{\lambda}\sum_{i=1}^{j}\ln(U_i) = \sum_{i=1}^{j} T_i \sim \text{Gamma}(j,\lambda).$$

For $n \geq 1$, observe that

$$\{X \geq n\} = \{\frac{-1}{\lambda}\ln(U_1U_2\ldots U_n) \leq 1\} = \{\sum_{i=1}^{n} T_i \leq 1\}$$

Using this, for $n \geq 1$,

$$\begin{aligned}\mathbb{P}(X=n) &= \mathbb{P}(X \geq n) - \mathbb{P}(X \geq n+1) \\ &= \mathbb{P}(\sum_{i=1}^{n} T_i \leq 1) - \mathbb{P}(\sum_{i=1}^{n+1} T_i \leq 1) \\ &= \frac{\lambda^n}{n-1!}\int_0^1 z^{n-1}e^{-\lambda z}dz - \frac{\lambda^{n+1}}{n!}\int_0^1 z^n e^{-\lambda z}dz \qquad \text{(Using the p.d.f of Gamma distribution)} \\ &= \sum_{m=n}^{\infty} e^{-\lambda}\frac{(\lambda)^m}{m!} - \sum_{m=n+1}^{\infty} e^{-\lambda}\frac{(\lambda)^m}{m!} \qquad \text{(Using the assumption given)} \\ &= e^{-\lambda}\frac{(\lambda)^n}{n!}. \qquad \text{(As the above of limit of partial sums)}\end{aligned} \tag{3}$$

From (1),(2), and (3) we conclude that $X \sim \text{Poisson}(\lambda)$ □

[1] i.e Range(X) and $\mathbb{P}(X = x)$ for all $x \in$ Range(X).