Question: Let X be a random variable with finite range $\{c_1, c_2\}$ for which $P(X = c_j) = p_j > 0$ for $j = 1, 2$. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample with distribution X and let $Y_j = |\{j : X_j = c_j\}|$. Define

$$X^2 = \sum_{j=1}^{2} \frac{(Y_j - np_j)^2}{np_j}.$$

Let $F_n : [0, \infty) \to [0, 1]$ be the distribution function of X^2.

Show that there is a differentiable $F : \mathbb{R} \to [0, 1]$ such that $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$ and find F'.

Answer: Observe that $n = Y_1 + Y_2$ and $1 = p_1 + p_2$. So we have

$$\begin{aligned}
X^2 &= \sum_{j=1}^{2} \frac{(Y_j - np_j)^2}{np_j}. \\
&= \frac{(Y_1 - np_1)^2}{np_1} + \frac{(Y_2 - np_2)^2}{np_2} \\
&= \frac{(Y_1 - np_1)^2}{np_1} + \frac{(n - Y_1 - n(1 - p_1)^2}{n(1 - p_1)} \\
&= \frac{(Y_1 - np_1)^2}{np_1(1 - p_1)}
\end{aligned}$$

By properties of Multinomial distribution, $Y_1 \sim \text{Binomial}(n, p_1)$. By the Central Limit Theorem, we know that

$$\frac{(Y_1 - np_1)}{\sqrt{np_1(1 - p_1)}} \xrightarrow{d} Z$$

where $Z \sim N(0, 1)$. Further using facts about convergence in distribution we have that

$$X^2 = \left(\frac{(Y_1 - np_1)}{\sqrt{np_1(1 - p_1)}} \right)^2 \xrightarrow{d} Z^2,$$

where $Z^2 \sim \Gamma(\frac{1}{2}, \frac{1}{2}) \equiv \chi_1^2$. Let F be the distribution function of Z^2. Since F is continuous, by the above we have

$$F_n(x) \to F(x) \text{ for all } x \in \mathbb{R}.$$

Further F' is the p.d.f of Z^2 which is given by

$$F'(x) = \begin{cases} \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

□