Question: Let $0 < \alpha, \beta$ and $X_1, X_2, \dots, X_m$ be i.i.d. random variables with commmon probability density function with Weibull (α, β). Assume $\beta > 0$ is a known constant. Find the maximum likelihood estimator for α.

Answer: We know that the common p.d.f of $X_1, X_2, \dots, X_m$ is given by

$$f(x \mid \alpha, \beta) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that $\alpha, \beta, X_1, X_2, \dots X_m$ are all positive. Then the Likelihood function from the sample $X_1, X_2, \dots, X_n$ is then given by

$$L(\alpha, \beta; X_1, X_2, \dots X_m) = \prod_{i=1}^{m} f(X_i \mid \alpha, \beta) = \alpha^m \beta^m \left(\prod_{i=1}^{m} X_i \right)^{\beta-1} e^{-\alpha \sum_{i=1}^{m} X_i^\beta}.$$

The log-Likelihood function is given by

$$LL(\alpha, \beta; X_1, X_2, \dots X_m) = m \ln(\alpha) + m \ln(\beta) + (\beta - 1) \sum_{i=1}^{m} \ln(X_i) + -\alpha \sum_{i=1}^{m} X_i^\beta.$$

For given $\beta > 0, X_1, X_2, \dots X_m > 0$ The above function is twice differentiable in α for all $\alpha > 0$ and its

$$\frac{\partial}{\partial \alpha} LL(\alpha, \beta; X_1, X_2, \dots X_m) = m \frac{1}{\alpha} - \sum_{i=1}^{m} X_i^\beta.$$

and

$$\frac{\partial^2}{\partial^2 \alpha} LL(\alpha, \beta; X_1, X_2, \dots X_m) = -m \frac{1}{\alpha^2} < 0.$$

Since the second derivative is always negative when $\alpha > 0$ and the firt derivative is zero at $\frac{m}{\sum_{i=1}^{m} X_i^\beta}$ we have that the M.L.E. for α is given by

$$\hat{\alpha} = \frac{m}{\sum_{i=1}^{m} X_i^\beta}.$$