Due Date: September 26th, 2019
Problems due: 2(c), 3(b), 4,5,6

1. Let $n \geq 1$ and let $X_1, X_2, \ldots, X_n$ be a i.i.d random sample from uniform $(0,1)$. Let the X's be arranged in increasing order of magnitude denoted by

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}.$$

These ordered values are called the order statistics of the sample $X_1, X_2, \ldots, X_n$. For, $1 \leq r \leq n$, $X_{(r)}$ is called the r-th order statistic. Find the probability density function of, $X_{(r)}$, the r-th order statistic for $1 \leq r \leq n$.

2. Suppose $X_1, X_2, \ldots, X_n$ be an i.i.d. random sample from a Normal mean 0 and variance 1 population and $Y_1, Y_2, \ldots, Y_m$ be an independent i.i.d. random sample from a Normal mean 0 and variance 1 population. Let

$$V = \sum_{i=1}^{m} Y_i^2 \quad \text{and} \quad U = \sum_{i=1}^{n} X_i^2$$

(a) (*Chi-Square with* n *degrees of freedom*) Show that U has χ_n^2 distribution, That is the probability density function of U is given by:

$$f(u) \quad = \quad \frac{2^{-\frac{n}{2}}}{\Gamma(\frac{n}{2})} u^{\frac{n}{2}-1} e^{-\frac{u}{2}} = \begin{cases} \frac{2^{-\frac{n}{2}}}{(\frac{n}{2}-1)!} u^{\frac{n}{2}-1} e^{-\frac{u}{2}} & \text{when } n \text{ is even,} \\ \\ \frac{2^{n-\frac{n}{2}-1}(\frac{n-1}{2})!}{(n-1)!\sqrt{\pi}} u^{\frac{n}{2}-1} e^{-\frac{u}{2}} & \text{when } n \text{ is odd,} \end{cases}$$

for $u \in \mathbb{R}$.

(b) (*F-distribution* $F(n,m)$) Let

$$Z = \frac{U/n}{V/m}.$$

Show that Z has $F(n,m)$ distribution, that is its probability density function for $z > 0$ is given by

$$f(z) = \left(\frac{m}{n}\right)^{\frac{n}{2}} \frac{z^{\frac{n}{2}-1}}{(1+\frac{n}{m}z)^{\frac{n+m}{2}}} \frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}.$$

(c) (t_m*-distribution*) Let $n = 1$,

$$W = \sqrt{Z} = \frac{X_1}{\sqrt{\frac{V}{m}}}.$$

Show that W has the t_m distribution. That is, its probability density function for $w \in \mathbb{R}$ is given by

$$f_W(w) \quad = \quad \frac{\Gamma(\frac{m+1}{2})}{\sqrt{m\pi}\Gamma(\frac{m}{2})} \left(1 + \frac{w^2}{m}\right)^{-\frac{m+1}{2}}$$

3. Assume that for given x-values $X_1, X_2, \ldots, X_n$ the corresponding y-values $Y_1, Y_2, \ldots, Y_n$ are given by

$$Y_j = \beta_0 + \beta_1 X_j + \epsilon_j, \tag{1}$$

for $j = 1, 2, \ldots, n$ and where each of the ϵ_j are independent random variables with $\epsilon_j \sim \text{Normal}(0, \sigma^2)$.

(a) Show that the least squares $\hat{\beta}_1$ has a normal distribution with mean β_1 and variance $\frac{\sigma^2}{(n-1)S_X^2}$.

(b) Show that $\hat{\beta}_1$ and $\bar{Y}$ are independent.

(c) Show that the least squares $\hat{\beta}_0$ has a normal distribution with mean β_0 and variance $\sigma^2(\frac{1}{n} + \frac{\overline{X}^2}{(n-1)S_X^2})$.

(d) Suppose RSS $= \sum_{i=1}^n \epsilon_i^2$. Find E[RSS] and Var[RSS]. Can you find the distribution of RSS ?

(e) Suppose for a particular deterministic x-value X^* that we want to use the data to estimate the corresponding y-value Y^* using the least square line. Find the distribution of Y^*.

4. Suppose Ruhi has a coin that Ruhi claim is "fair" (equally likely to come up heads or tails) and that her friend claims is weighted towards heads. Suppose Ruhi flip the coin twenty times and find that it comes up heads on sixteen of those twenty flips. While this seems to favor her friend's hypothesis, it is still possible that Ruhi is correct about the coin and that just by chance the coin happened to come up heads more often than tails on this series of flips. Let S be the sample space of all possible sequences of flips. The size of S is then 2^{20}, and if Ruhi is correct about the coin being "fair", each of these outcomes are equally likely.

   (a) Let E be the event that exactly sixteen of the flips come up heads. What is the size of E? What is the probability of E?

   (b) Let F be the event that at least sixteen of the flips come up heads. What is the size of F? What is the probability of F?

5. Two types of coin are produced at a factory: a fair coin and a biased one that comes up heads 55% of the time. We have one of these coins but do not know whether it is a fair or biased coin. In order to ascertain which type of coin we have, we shall perform the following statistical test. We shall toss the coin 1000 times. If the coin comes up heads 525 or more times we shall conclude that it is a biased coin. Otherwise, we shall conclude that it is fair. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

6. The length of time (in appropriate units) that a certain type of component functions before failing is a random variable with probability density function

$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Once the component fails it is immediately replaced with another one of the same type. Using the central limit theorem approximation, can you find, how many components would one need to have on hand to be approximately 90% certain that the stock would last at least 35 units of time ?