Ground Rules :-

- Siva [Please do not call me Sio]

- Zoom Room [· 1:55 Thursday – open
  - · 2:05 – 2:45 pm Part I
  - · 2:50 – 3:30 pm Part II*
  - · 3:35 pm – close]

- Syllabus & other features / Evaluations
  - See the Piazza page
  - Copy of documents on Moodle
    - on website.

- Take home assignments. [2 in a month]

- Backgrounds :- Diverse
  - Assumption on background in Probability ← let me know

Yogesh :- Focus on Sparse graph & its limits

Here :- Focus will be on dense graphs

Introduction to Course :-

$$V = \{1, \dots, n\} \qquad n \in \mathbb{N}$$

$$E_n \subseteq \{ \{i,j\} : 1 \le i \le n,\ 1 \le j \le n,\ i \ne j \}$$

$$G_n = (V, E_n)$$

$$e_n = \#\text{ of edges in } E_n = |E_n|$$

- $\{G_n\}_{n \ge 1} \equiv$ sequence of graphs is said to be a dense graph sequence

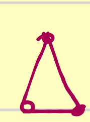
if $\displaystyle \liminf_{n \to \infty} \frac{e_n}{n^2} > 0$ "e_n is of order n^2".

- F is a finite graph on k-vertices $(k \le n)$

$$0 \le t(F, G_n) := \frac{\#\text{ of copies of } F \text{ in } G_n}{\#\text{ of copies of } F \text{ in } K_n} \le 1$$

where K_n is the complete graph on V

<u>Convention</u> :- $k > n$ $\quad t(F, G_n) = 0$.

<u>Example</u> :- $k=2$ $\quad F_2$  $\quad F_3$ 

$$t(F_2, G_n) \equiv \text{edge density in } G_n$$

$$t(F_3, G_n) \equiv \text{triangle density in } G_n$$

Fix an enumeration of $\{F_i\}_{i\geq 1}$ of finite graphs.

Say G, H are two graphs on V, V'

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \left| t(F_i, G) - t(F_i, H) \right| := d_{sub}(G, H)$$

"sub" = Subgraph Count metric

d_{sub} Metric :- on isomorphism classes

Question :- $\{G_n\}_{n\geq 1}$ - a sequence of dense graphs, Say under d_{sub} G_n - is a cauchy sequence. Does G_n converge to "?" in d_{sub} metric?

- $K : [0,1]^2 \to [0,1]$ Symmetric and measurable [Graphon]

F - finite graph on m-vertices

$$t(F, K) = \int \prod_{i=1}^{m} dx_i \prod_{(i,j)\in E(F)} K(x_i, x_j)$$

$$= \mathbb{E} \prod_{(i,j)\in E(F)} K(U_i, U_j)$$

where $U_i \sim \text{Uniform}[0,1]$ and independent

Theorem :- [Lovasz - Szegedy 2006, Diaconis - Janson 2008]

If $\{G_n\}_{n\geq 1}$ is a dense graph sequence & is a Cauchy sequence under d_{sub} then $\exists$ a graphon $K : [0,1]^2 \to [0,1]$ such that

$d_{sub}(G_n, K) \to 0$ as $n \to \infty$.

First 3/4 of the course :- will be making all def., terms, statements, precise and provide a proof

Construct finite graph from Graphons

$K : [0,1]^2 \to [0,1]$

$V = \{1, \dots, n\}$

$\uparrow \quad \uparrow$

$U_1 \; . \; U_n$ – uniform i.i.d labels $U(0,1)$

$E :=$ $i \sim j$ w.p. $K(U_i, U_j)$

– $G(n, K)$ — Theorem $\Leftrightarrow$ SLLN

Examples :-

(I) $K(x,y) = p \quad \forall \; 0 \le x \le y \le 1$

$0 \le p \le 1$

$V = \{1, \dots n\}$

$i \sim j$ w.p. p

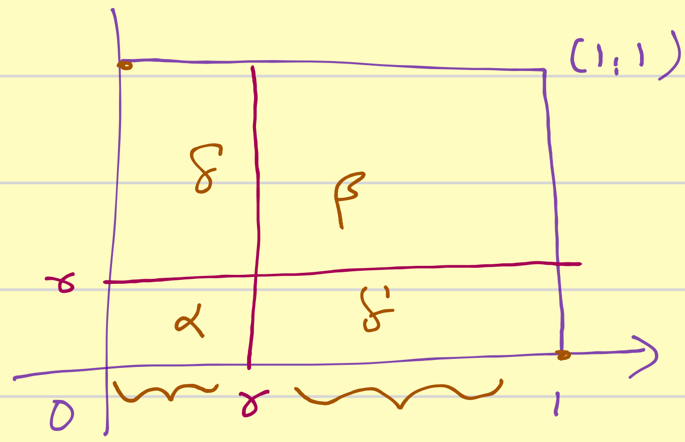
– Erdös Renyi Graph – $G(n,p)$

F – graph on m – vertices

$t(F, G(n,p)) = \dots$

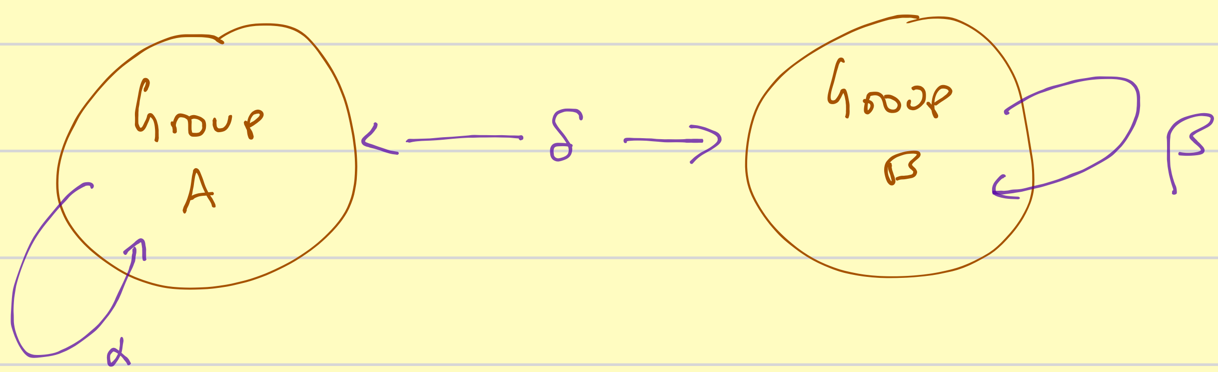
Exercise :- $\xrightarrow[\text{SLLN}]{n \to \infty}$ $t(F, K)$

(II) $K : [0,1]^2 \to [0,1]$



• $1, \dots, n$ $\quad i \sim j$ w.p. $K(u_i, u_j)$

$u_1, \dots, u_n$



(III) (Deterministic Graph)

$$\mathcal{G} = \begin{cases} V = \{1, \dots, n\} \\ E \subseteq \{\{i,j\} \mid 1 \le i \le n,\ 1 \le j \le n\} \end{cases}$$

$K_{\mathcal{G}} : [0,1]^2 \to [0,1]$

$$K_{\mathcal{G}}(x,y) = \begin{cases} 1 & \text{if } \lfloor nx \rfloor \sim \lfloor ny \rfloor \in E \\ 0 & \text{otherwise} \end{cases}$$

(Graphon)