1. Let $Y_1,\ Y_2,\ \cdots,\ Y_n$ be independent random variables, each uniformly distributed over the interval $(0, \theta)$.

(a) Show that the mean $\bar{Y}$ converges in probability towards a constant as $n \to \infty$ and find the constant.

(b) Show that $\max\{Y_1, \cdots, Y_n\}$ converges in probability toward θ as $n \to \infty$.

Solution: (a) Each Y_i has mean $\frac{\theta}{2}$ and variance $\frac{1}{12}\theta^2 < \infty$. Hence, by the Weak Law of Large Numbers, $\bar{Y}$ converges in probability to $\frac{\theta}{2}$.

(b) Call $M_n = \max\{Y_1, \cdots, Y_n\}$. Then, for any $\epsilon > 0$,

$$P\big(|M_n - \theta| \geq \epsilon\big) = P\big(M_n \leq \theta - \epsilon\big) = P\big(Y_1 \leq \theta - \epsilon,\ \cdots,\ Y_n \leq \theta - \epsilon\big) = \big(\frac{\theta - \epsilon}{\theta}\big)^n$$

so that

$$\lim_{n\to\infty} P\big(|M_n - \theta| \geq \epsilon\big) = \lim_{n\to\infty} \big(1 - \frac{\epsilon}{\theta}\big)^n = 0$$

or, equivalently,

$$\lim_{n\to\infty} P\big(|M_n - \theta| \leq \epsilon\big) = 1$$

This is the definition of "$\max\{Y_1, \cdots, Y_n\}$ converges in probability toward θ as $n \to \infty$".