Ground Rules: Time allowed is 15 minutes, individual work only and closed book test. You may work with class XII knowledge with regard to integtation.

1. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that f is a probability density function.

(ii) If I, J are two bounded intervals of $\mathbb{R}$, that have the same length, then is it true that $\mathbb{P}(I) = \mathbb{P}(J)$?

Solution: (i) By definition, $f(x) \geq 0$ for all $x \in \mathbb{R}$. Note that

$$f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & \text{if } a < x < b \\ 0 & x > b \end{cases}$$

Therefore it is continuous in the intervals $(-\infty, a), (a, b)$ and (b, ∞). The only discontinuity points are at $x = a$ and $x = b$. Therefore the function f is piecewise continuous. Finally,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} 0\, dx + \int_{a}^{b} \frac{1}{b-a}\, dx + \int_{b}^{\infty} 0\, dx = \frac{x}{b-a}\Big|_a^b = \frac{b}{b-a} - \frac{a}{b-a} = 1.$$

(ii) Consider the points

$$a - \frac{b-a}{4} < a + \frac{b-a}{4} < \frac{a+b}{2} < b.$$

Let $I = [a - \frac{b-a}{4}, a + \frac{b-a}{4}]$ and $J = [\frac{a+b}{2}, b]$. They are both bounded intervals of $\mathbb{R}$ having the same length $= \frac{b-a}{2}$. However,

$$\begin{aligned} \mathbb{P}(I) &= \int_I f(x)dx = \int_{a-\frac{b-a}{4}}^{a+\frac{b-a}{4}} f(x)dx \\ &= \int_{a-\frac{b-a}{4}}^{a} 0\, dx + \int_{a}^{a+\frac{b-a}{4}} \frac{1}{b-a}dx \\ &= \frac{x}{b-a}\Big|_a^{a+\frac{b-a}{4}} \\ &= \frac{a+\frac{b-a}{4}}{b-a} - \frac{a}{b-a} \\ &= \frac{b-a}{4} \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}(J) &= \int_J f(x)dx = \int_{\frac{a+b}{2}}^{b} f(x)dx \\ &= \int_{\frac{a+b}{2}}^{b} \frac{1}{b-a}dx \\ &= \frac{x}{b-a}\Big|_{\frac{a+b}{2}}^{b} \\ &= \frac{b}{b-a} - \frac{\frac{a+b}{2}}{b-a} \\ &= \frac{b-a}{2}. \end{aligned}$$

Thus $\mathbb{P}(I) \neq \mathbb{P}(J)$. So it is not true in general[1].

$\square$

[1] We note that if $I, J \subset [a, b]$ having the same length then $\mathbb{P}(I) = \mathbb{P}(J)$