Ground Rules: Time allowed is 15 minutes, individual work only and closed book test. You may assume any required fact from Real Analysis.

1. (25 points) Suppose that the number of earthquakes that occur in a year, in Soukarpet, has Poisson(λ) distribution. Further, suppose that the probability that any given earthquake has magnitude at least 5 on the Richter scale is p. Find the probability that there are 10 earthquakes with magnitude at least 5 in a year.

Solution: Let X be the number of earthquakes in a year in Soukarpet and B be the number of earthquakes in a year with magnitude at least 5. Then for $x \in \{0\} \cup \mathbb{N}$ we have

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}.$$

For $n \in \{0\} \cup \mathbb{N}$

$$P(B = n|X = x) \text{ is zero if } x < n, \text{ and}$$
$$P(B = n|X = x) = \binom{x}{n} p^n (1-p)^{x-n} \text{ otherwise.}$$

Hence for $n \geq 0$,

$$P(B = n) = P\Big(\bigcup_{x=0}^{\infty} (B = n) \cap (X = x)\Big)\Big) = \sum_{x=0}^{\infty} P\big((B = n) \cap (X = x)\big) \tag{1}$$

For $k \geq 0$, let $T_k = \sum_{x=0}^{k} P\big((B = n) \cap (X = x)\big)$. Now,

$$T_k = \sum_{x=0}^{k} P\big((B = n) \cap (X = x)\big) = \sum_{x=0}^{k} P(B = n|X = x)P(X = x).$$

So from the above $T_k = 0$ for $k < n$. For $k \geq n$

$$\begin{aligned} T_k &= \sum_{x=0}^{k} \binom{x}{n} p^n (1-p)^{x-n} e^{-\lambda} \frac{\lambda^x}{x!} \\ &= e^{-\lambda} p^n \sum_{x=n}^{k} \frac{\lambda^x}{n!(x-n)!}(1-p)^{x-n} \\ &= \frac{1}{n!} e^{-\lambda} (\lambda p)^n \sum_{m=0}^{k-n} \frac{\lambda^m}{m!}(1-p)^m, \end{aligned} \tag{2}$$

where in the last line we changed $m = x - n$. From Analysis I: we have

$$\lim_{k \to \infty} \sum_{m=0}^{k-n} \frac{(\lambda(1-p))^m}{m!} = e^{-\lambda(1-p)}. \tag{3}$$

Using the (1), (2) and (3), we have

$$P(B=n)=\frac{1}{n!}e^{-\lambda}(\lambda p)^n\ e^{\lambda(1-p)}=e^{-\lambda p}\frac{(\lambda p)^n}{n!}$$

So,

$$\begin{aligned}
&P(\text{there are 10 earthquakes with magnitude at least 5 in a year. })\\
&=P(B=10)\\
&=e^{-\lambda p}\frac{(\lambda p)^{10}}{10!}
\end{aligned}$$