Recall:-

$X \sim \text{Normal}(\mu, \sigma^2)$ $\quad \mu \in \mathbb{R}, \ \sigma > 0$

- if X has p.d.f given by

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \qquad x \in \mathbb{R}$$

- $Z \sim \text{Normal}(0,1)$ if Z has p.d.f given by

[Standard Normal]

$$f(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \qquad z \in \mathbb{R}$$

- [De Moivre-Laplace Central limit Theorem]

$S_n \sim \text{Binomial}(n,p)$ $\quad 0 < p < 1$

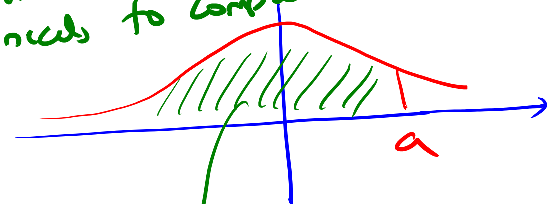
$a, b \in \mathbb{R}$, $a < b$

$$\mathbb{P}\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} < b\right) \xrightarrow[\text{as } n \to \infty]{} \int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$$\left[\ p \sim \frac{\lambda}{n} \ - \ \mathbb{P}(S_n = k) \xrightarrow[\text{as } n\to\infty]{} \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{Poisson}\ \right]$$

- $$\mathbb{P}(Z \le a) = \int_{-\infty}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

. no antiderivative
. one needs to compute numerically



$\mathbb{P}(Z \le a)$

Symmetry Property of Probability density functions

- $\mathbb{P}(Z \le 0) = \mathbb{P}(Z \ge 0) = \frac{1}{2}$

$\underline{a < 0}$

- $\mathbb{P}(Z \le a) = 1 - \mathbb{P}(Z \le -a)$

$\underline{a > 0}$

- $$\mathbb{P}(Z \le a) = \frac{1}{2} + \int_0^a \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$X \sim \text{Normal}(\mu, \sigma^2)$

$$\mathbb{P}(X \leq b) = \int_{-\infty}^{b} f(x)\, dx = \int_{-\infty}^{b} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\, dx$$

[Substitution] $z = \frac{x-\mu}{\sigma}$, $dz = \frac{1}{\sigma} dx$

$$= \int_{-\infty}^{\frac{b-\mu}{\sigma}} \frac{e^{-\frac{z^2}{2}}}{\cancel{\sigma}\sqrt{2\pi}}\, \cancel{\sigma}\, dz$$

$$= \int_{-\infty}^{\frac{b-\mu}{\sigma}} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$$\therefore \mathbb{P}(X \leq b) = \mathbb{P}\left(Z \leq \frac{b-\mu}{\sigma}\right)$$

where $Z \sim \text{Normal}(0,1)$

- To calculate probabilities of $X \sim \text{Normal}(\mu, \sigma^2)$; it is enough to calculate / Tabulate numerically the probabilities of $Z \sim \text{Normal}(0,1)$.

- Normal Tables: (Chapter 5 - page 128 - Book)

	0.00	0.02	0.04	0.06	$\boxed{0.08}$	0.10	0.12	0.14	0.16	0.18
0.0	0.500	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564	0.571
0.2	0.579	0.587	0.595	0.603	0.610	0.618	0.626	0.633	0.641	0.648
0.4	0.655	0.663	0.670	0.677	0.684	0.691	0.698	0.705	0.712	0.719
0.6	0.726	0.732	0.739	0.745	0.752	0.758	0.764	0.770	0.776	0.782
0.8	0.788	0.794	0.800	0.805	0.811	0.816	0.821	0.826	0.831	0.836
1.0	0.841	0.846	0.851	0.855	0.860	0.864	0.869	0.873	0.877	0.881
$\boxed{1.2}$	0.885	0.889	0.893	0.896	$\boxed{0.900}$	0.903	0.907	0.910	0.913	0.916
1.4	0.919	0.922	0.925	0.928	0.931	0.933	0.936	0.938	0.941	0.943
1.6	0.945	0.947	0.949	0.952	0.954	0.955	0.957	0.959	0.961	0.962
1.8	0.964	0.966	0.967	0.969	0.970	0.971	0.973	0.974	0.975	0.976
2.0	0.977	0.978	0.979	0.980	0.981	0.982	0.983	0.984	0.985	0.985

Table 5.1: Table of Normal(0, 1) probabilities. For $X \sim$ Normal(0, 1), the table gives values of $P(X \leq z)$ for various values of z between 0 and 2.18 upto three digits. The value of z for each entry is obtained by adding the corresponding row and column labels.

$$\int_{-\infty}^{1.28} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz = \mathbb{P}(X < 1.2 + 0.08) \cong 0.9$$

Example 5.2.12 A machine fills bags with cashews

- intended weight of each bag = 200 gm
- machine has tolerance ≡ Normal distributed with $\mu = 200$ and $\sigma = 4$

Question: How likely is it that a bag filled by the machine will have fewer than 195 gms?

Answer:- $Y \sim \text{Normal}(200, 16)$

$$\mathbb{P}(Y \le 195) = \mathbb{P}\left(Z \le \frac{195 - 200}{4}\right)$$

$$= \mathbb{P}(Z \le -5/4)$$

Symmetry calculation $\leftarrow$ $= 1 - \mathbb{P}(Z \le 5/4)$

$$= 1 - \mathbb{P}(Z \le 1.25)$$

	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
0.0	0.500	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564	0.571
0.2	0.579	0.587	0.595	0.603	0.610	0.618	0.626	0.633	0.641	0.648
0.4	0.655	0.663	0.670	0.677	0.684	0.691	0.698	0.705	0.712	0.719
0.6	0.726	0.732	0.739	0.745	0.752	0.758	0.764	0.770	0.776	0.782
0.8	0.788	0.794	0.800	0.805	0.811	0.816	0.821	0.826	0.831	0.836
1.0	0.841	0.846	0.851	0.855	0.860	0.864	0.869	0.873	0.877	0.881
1.2	0.885	0.889	0.893	$\boxed{0.896}$	0.900	0.903	0.907	0.910	0.913	0.916
1.4	0.919	0.922	0.925	0.928	0.931	0.933	0.936	0.938	0.941	0.943
1.6	0.945	0.947	0.949	0.952	0.954	0.955	0.957	0.959	0.961	0.962
1.8	0.964	0.966	0.967	0.969	0.970	0.971	0.973	0.974	0.975	0.976
2.0	0.977	0.978	0.979	0.980	0.981	0.982	0.983	0.984	0.985	0.985

Table 5.1: Table of Normal$(0,1)$ probabilities. For $X \sim$ Normal$(0,1)$, the table gives values of $P(X \le z)$ for various values of z between 0 and 2.18 upto three digits. The value of z for each entry is obtained by adding the corresponding row and column labels.

$$\cong 1 - \mathbb{P}(Z \le 1.26)$$

$$= 1 - 0.896$$

$$= 0.104 \approx 10\%$$

- $X \sim \text{Normal}(\mu, \sigma^2)$ $\quad \mu = E[X] \quad \sigma = SD[X]$ [Kept referring to this!]

Example 6.1.11 :-

$$Z \sim \text{Normal}(0,1)$$

$$E[Z] = \int_{-\infty}^{\infty} z \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz$$

i.e $\int_{-\infty}^{\infty} \left| z \frac{e^{-z^2/2}}{\sqrt{2\pi}} \right| dz < \infty$? — integral converges absolutely

$\boxed{\text{Analysis I Calculus}}$ $\exists c_1 > 0 : \quad z e^{-z^2/2} \le e^{-c_1 z} \quad \forall z > 0$

$$\int_{-\infty}^{\infty} \left| z \frac{e^{-z^2/2}}{\sqrt{2\pi}} \right| dz = 2 \int_{0}^{\infty} z \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz$$

$$\le \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-c_1 z} \, dz$$

$$= \frac{2}{\sqrt{2\pi}} \left. \frac{e^{-c_1 z}}{-c_1} \right|_{0}^{\infty}$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{1}{c_1} - 0 \right) < \infty$$

$$E[Z] = \int_{-\infty}^{\infty} \frac{x e^{-x^2/2}}{\sqrt{2\pi}} \, dx = \int_{-\infty}^{0} \frac{x e^{-x^2/2}}{\sqrt{2\pi}} \, dx + \int_{0}^{\infty} \frac{x e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$

$$\underset{dy = -dx}{\overset{y = -x}{=}} \cancel{-\int_{0}^{\infty} \frac{y e^{-y^2/2}}{\sqrt{2\pi}} \, dy} + \cancel{\int_{0}^{\infty} \frac{x e^{-x^2/2}}{\sqrt{2\pi}} \, dx}$$

$$= 0$$

$$\mathrm{Var}[Z] = E[Z - E[Z]]^2 = E[Z^2]$$

$$= \int_{-\infty}^{\infty} \frac{z^2 e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

Calculus: $\exists c_2 > 0$: $z^2 e^{-\frac{z^2}{2}} \le e^{-c_2 z} \quad \forall z > 0$.

$$= \int_{-\infty}^{0} \frac{z^2 e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz + \int_{0}^{\infty} \frac{z^2 e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

$y = -z$, $dy = dz$

$$= \int_{0}^{\infty} \frac{y^2 e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy + \int_{0}^{\infty} \frac{z^2 e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

$$= 2 \int_{0}^{\infty} \frac{z^2 e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} z\, z e^{-\frac{z^2}{2}}\, dz$$

Integration by parts:-

$u = z \qquad v = e^{-\frac{z^2}{2}}$

$u' = 1 \qquad v' = -z e^{-z^2/2}$

$$E[Z^2] = \frac{2}{\sqrt{2\pi}} \left[u(-v) \Big|_0^{\infty} - \int_0^{\infty} u'(z)(-v(z))\, dz \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[0 - \int_0^{\infty} -e^{-z^2/2}\, dz \right]$$

$$= 2 \int_0^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

$$= 2\, \mathbb{P}(Z \le 0) = 2 \cdot \frac{1}{2} = 1$$

$\therefore \mathrm{Var}[Z] = 1$

$$\boxed{Z \sim \mathrm{Normal}(0,1) \qquad \mu = E[Z] = 0, \quad \sigma^2 = \mathrm{Var}[Z] = 1}$$

$Z \sim \text{Normal}(0,1)$ $\quad a, b \in \mathbb{R}$ $\quad X = aZ + b$

- linearity of Expectation $\quad E[X] = a\,E[Z] + b$

  $E[X] = b$

- Properties of variance $\quad Var[X] = a^2 Var[Z]$

  $Var[X] = a^2$

Distribution of X :- $\quad \underline{a > 0}$

$$\mathbb{P}(X \le x) = \mathbb{P}(aZ + b \le x)$$

$$= \mathbb{P}\left(Z \le \frac{x-b}{a}\right)$$

$$= \int_{-\infty}^{\frac{x-b}{a}} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$z = \frac{y-b}{a}$

$dz = \frac{dy}{a}$

$$= \int_{-\infty}^{x} \frac{e^{-\frac{(y-b)^2}{2a^2}}}{\sqrt{2\pi}}\, \frac{dy}{a}$$

$$= \int_{-\infty}^{x} f(y)\, dy$$

$$f(y) = \underbrace{\frac{e^{-\frac{(y-b)^2}{2a^2}}}{a\sqrt{2\pi}}}_{\text{p.d.f of Normal}(b,a^2)}$$

$|||$

Probability Normal with parameters a and b^2 $\le x$

$\therefore$ $Z \sim \text{Normal}(0,1)$ then $X = aZ + b$ is $\text{Normal}(a, b^2)$

By our earlier calculation we showed that $E[X] = a$ and $\text{var}[X] = b^2$ $\square$

<u>Transformation</u> $Z \sim \text{Normal}(0,1)$

$Y = Z^2$

<u>$y \geq 0$</u>

$$\mathbb{P}(Y \leq y) = \mathbb{P}(Z^2 \leq y)$$

$$= \mathbb{P}(-\sqrt{y} \leq Z \leq \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz$$

$$= 2 \int_{0}^{\sqrt{y}} \underbrace{\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}}_{\text{continuous}}\, dz$$

$$f_Y(y) = F_Y'(y) = 2 \cdot \frac{1}{2} y^{-\frac{1}{2}} \cdot \frac{e^{-y/2}}{\sqrt{2\pi}} \qquad y \geq 0$$

$$= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-y/2} \qquad y \geq 0$$

$y \leq 0 \qquad \mathbb{P}(Y \leq y) = \mathbb{P}(Z^2 \leq y) = 0$

$$f_Y(y) = 0 \qquad y \leq 0$$

$Y \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$$f(y) = \begin{cases} \frac{1}{\sqrt{2\pi}}\, y^{-1/2} e^{-y/2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Gamma distribution -