Recall :-

$X \sim \text{Binomial}(n, p)$

- $\mathbb{P}(X=k) = \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}$ $\quad k \in \{0, 1, \dots, n\}$

$\boxed{\text{large } n \quad \text{"} k \approx \sqrt{n} \text{"} \quad \text{Computation issues}}$

- $p = \frac{\lambda}{n}$ $\quad \lim_{n \to \infty} \mathbb{P}(X=k) = \frac{e^{-\lambda} \lambda^k}{k!}$

(Poisson - approximation)

- $X_i \sim \text{Bernoulli}(p)$

$X = \sum_{i=1}^{n} X_i$

- $Y \sim \text{Poisson}(\lambda)$

$\mathbb{P}(Y=k) = \frac{e^{-\lambda} \lambda^k}{k!}$

$k \in \mathbb{N} \cup \{0\}$

Question: If $0 < p < \frac{1}{2}$ $\left(p \simeq \frac{1}{2}\right)$ fixed.

What is the behavior of X as n - gets larger?

Rewrite notation :

$X = \sum_{i=1}^{n} X_i \qquad X_i \sim \text{Bernoulli}(p)$

let us denote X as $S_n = \sum_{i=1}^{n} X_i$ $\quad$ [Behaviour as $n \to \infty$?]

$a, b \in \mathbb{R} \quad a < b \qquad \mathbb{P}(a \le S_n < b) = ?$ $\quad$ [as and $n \to \infty$]

[change with n?] - Broad signatures are :-

$E[S_n] = np$, $\quad Var(S_n) = Var\left(\sum_{i=1}^{n} X_i\right)$

$\overset{\text{independent}}{=} \sum_{i=1}^{n} Var(X_i)$

$= np(1-p)$

"Range of S_n" $\simeq$ "$\left(-\sqrt{np(1-p)} + np \, , \, np + \sqrt{np(1-p)}\right)$"

$SD(S_n)$

Modified Question: $a, b \in \mathbb{R}$ $a < b$

$$\mathbb{P}\left(a\sqrt{np(1-p)} + np \le S_n \le np + b\sqrt{np(1-p)} \right) = ?$$

ie $$\mathbb{P}\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b \right) = ?$$

Answer:-

$$\mathbb{P}\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b \right)$$

$$= \sum_{k \in A_n} \mathbb{P}(S_n = k)$$

where $A_n = \{ k \in \mathbb{N} \cup \{0\} : np + a\sqrt{np(1-p)} \le k \le np + b\sqrt{np(1-p)} \}$

$$= \sum_{k \in A_n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k \in A_n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$=$ Stirling's formula "$n! \approx n^n e^{-n} \sqrt{n} \dots$" $+$ Analysis II : Riemann integration "$\frac{1}{n} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \longrightarrow \int_0^1 f(x)\,dx$"

$$\xrightarrow[n \to \infty]{} \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$$

[De Moivre-Laplace Central limit Theorem]

Normal Random Variable :-

$Z \sim \text{Normal}(0,1)$ if Z has p.d.f given by

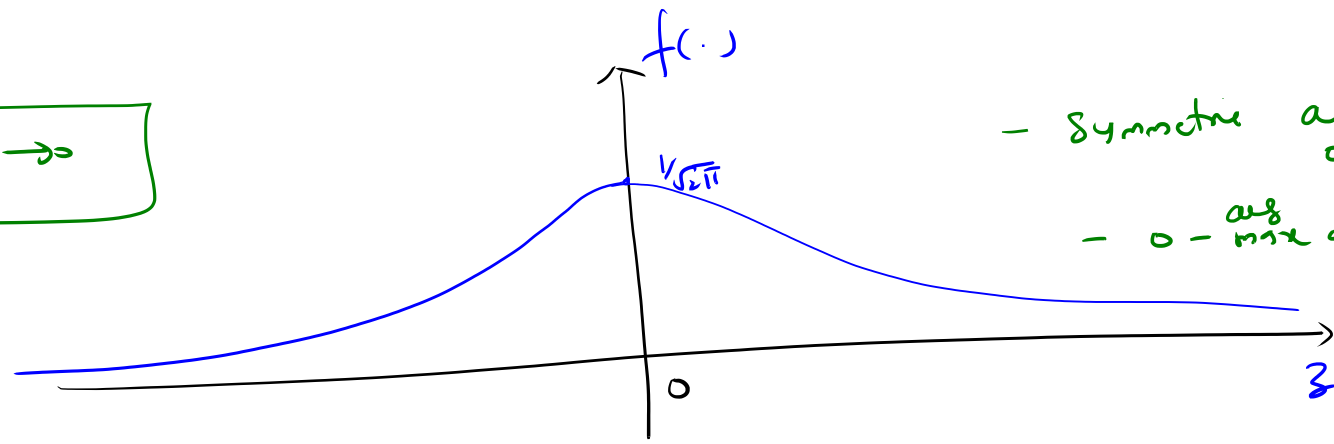
$$f(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \qquad z \in \mathbb{R}$$

Q0 :- Is f a density function?

A • $-\ f \geq 0$, f - continuous ✓ $\int_{-\infty}^{\infty} f(x)\,dx = 1$ [Assume - Analysis III - Complex Analysis]

O1 •

$\boxed{-\ z \to \pm\infty \quad f(\cdot) \to 0}$



- Symmetric around 0
- 0 - arg max of f

O2 • There is no anti derivative of $f(\cdot)$

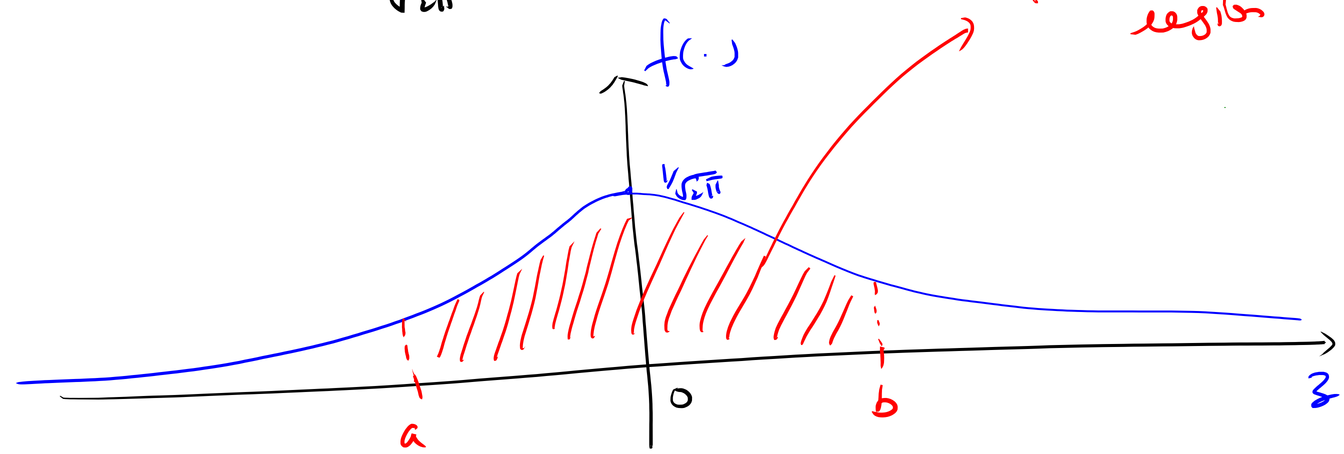
$$\therefore \int_a^b f(z)\,dz = \int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \qquad a, b \in \mathbb{R},\ a < b$$

can't be evaluated the standard $\overline{\underline{XII}}$ th class way.

(There are numerical tables called Normal tables that provide evaluations.)

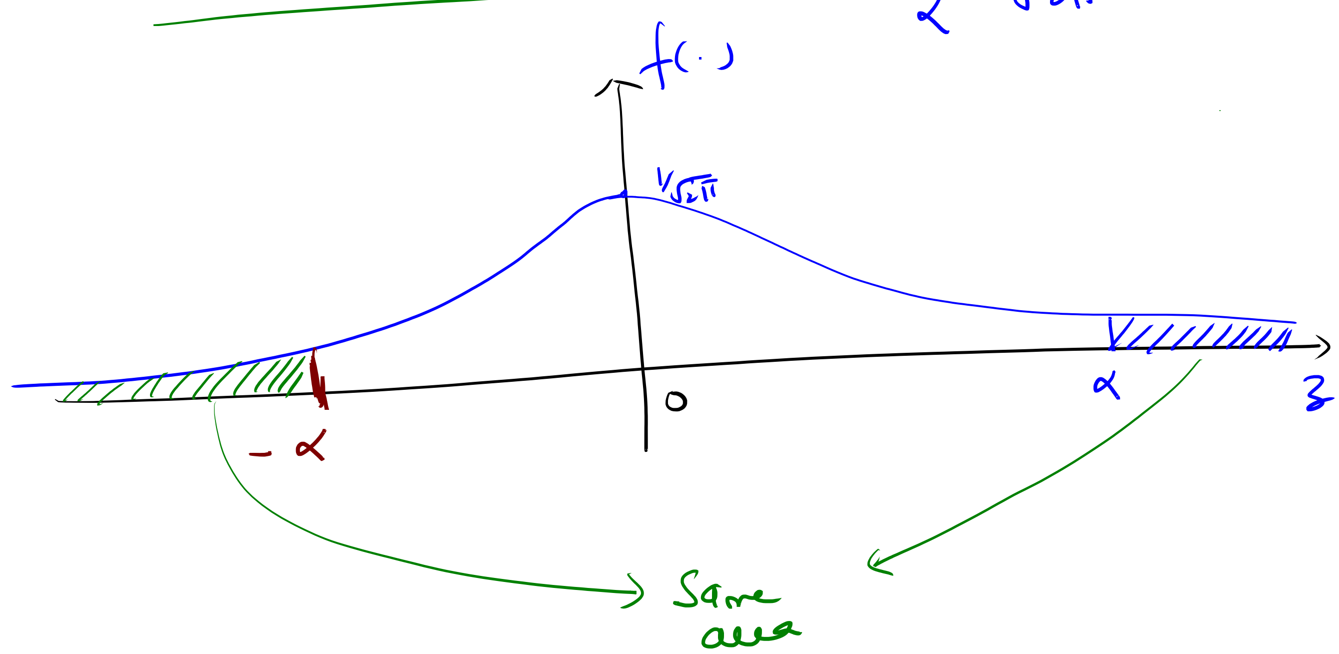
Q3 :- How does one compute Probabilities associated with Z ?

$$\mathbb{P}(a \le Z \le b) = \int_a^b \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz \quad \overset{\text{Analysis II}}{=\!=\!=} \quad \text{areas in this region}$$



Q1 :- $\displaystyle \int_{-\infty}^{-\alpha} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz \quad \overset{\substack{\text{Substitution} \\ u=-z} \,/\, \text{Analysis II}}{=\!=\!=\!=\!=} \quad \int_{\alpha}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$

$\alpha > 0$



$$\int_{-\infty}^{-\alpha} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz = \mathbb{P}(Z \le -\alpha)$$

$$1 - \mathbb{P}(Z \le \alpha) = 1 - \int_{-\infty}^{\alpha} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz = \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz - \int_{-\infty}^{\alpha} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz = \int_{\alpha}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$\quad ||$

$\mathbb{P}(Z > \alpha)$

$$\boxed{\mathbb{P}(Z \le -\alpha) = 1 - \mathbb{P}(Z \le \alpha)}$$

$$\mathbb{P}(a \le Z \le b) = \int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$$= \int_{-\infty}^{b} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz - \int_{-\infty}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

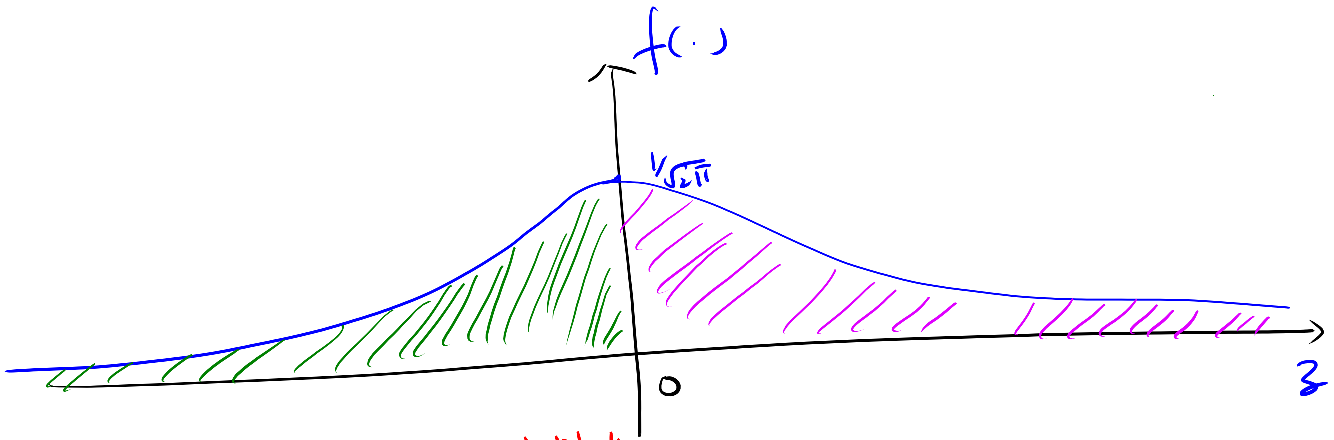
$\boxed{\text{Table required for all } a \,\&\, b}$

Claim:- It is enough to have a table of numerical values of: $\int_0^{\alpha} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$, $\alpha > 0$, to compute

$$\mathbb{P}(a \le Z \le b) \text{ for } a < b, \quad a, b \in \mathbb{R}.$$

Proof:-

Q2 •



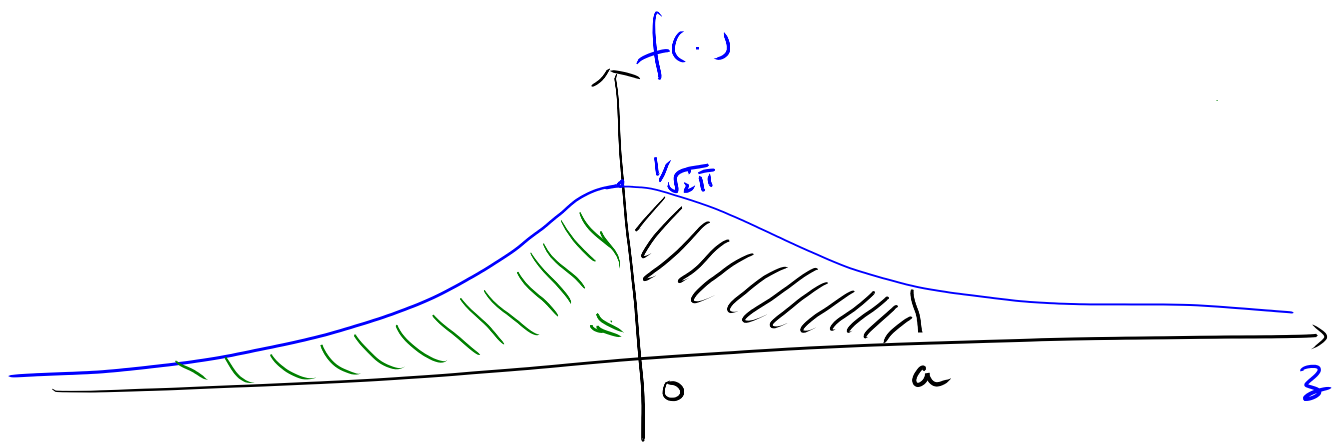
$$\frac{1}{2} = \int_{-\infty}^{0} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz \xrightarrow[\text{Analysis II}]{\text{Substitution } u=-z} \int_0^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz = \frac{1}{2}$$

$\_\_\_\_\_\_ \; (*) \; \_\_\_\_\_\_$

$$\int_{-\infty}^{0} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz + \int_{0}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = 1 \qquad -(*)$$

$\therefore$ • $a \geq 0$

$$\int_{-\infty}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = \int_{-\infty}^{0} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz + \int_{0}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$



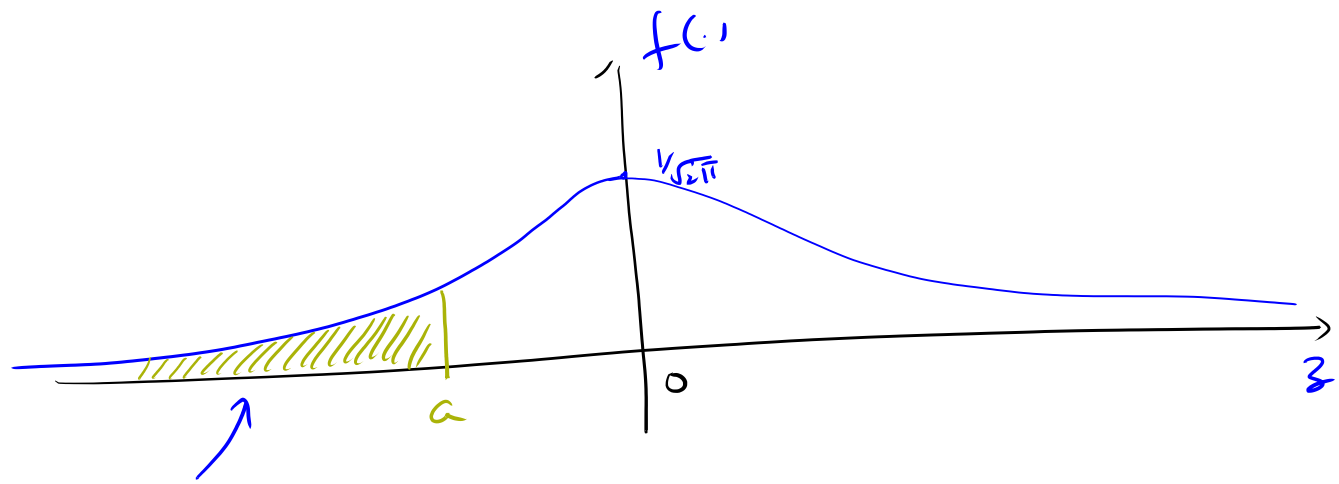
$a \geq 0$

$$\int_{-\infty}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = \frac{1}{2} + \underbrace{\int_{0}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz}_{\text{It is enough to tabulate such integrals}} \qquad -(+)$$

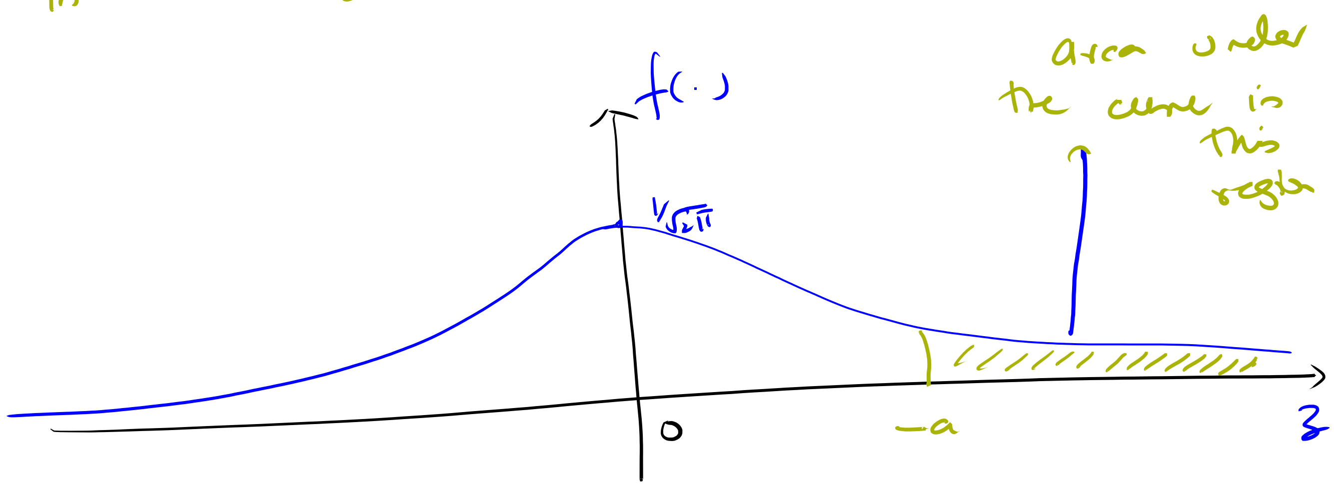
(1)

$a < 0$

$$\int_{-\infty}^{a} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz =$$

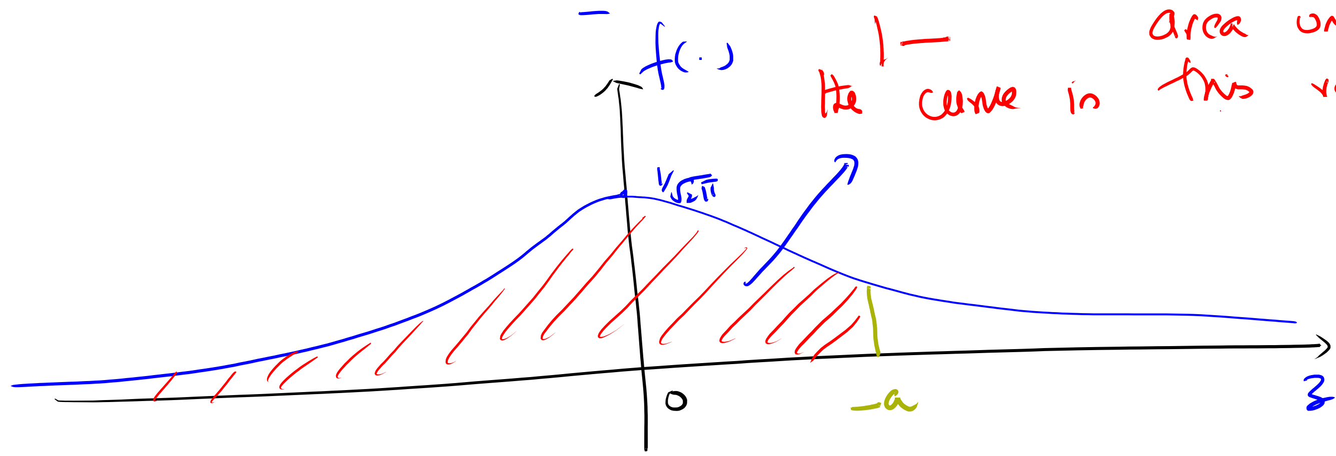


area under the curve in this region $=$



$=$ 1 − area under the curve in this region

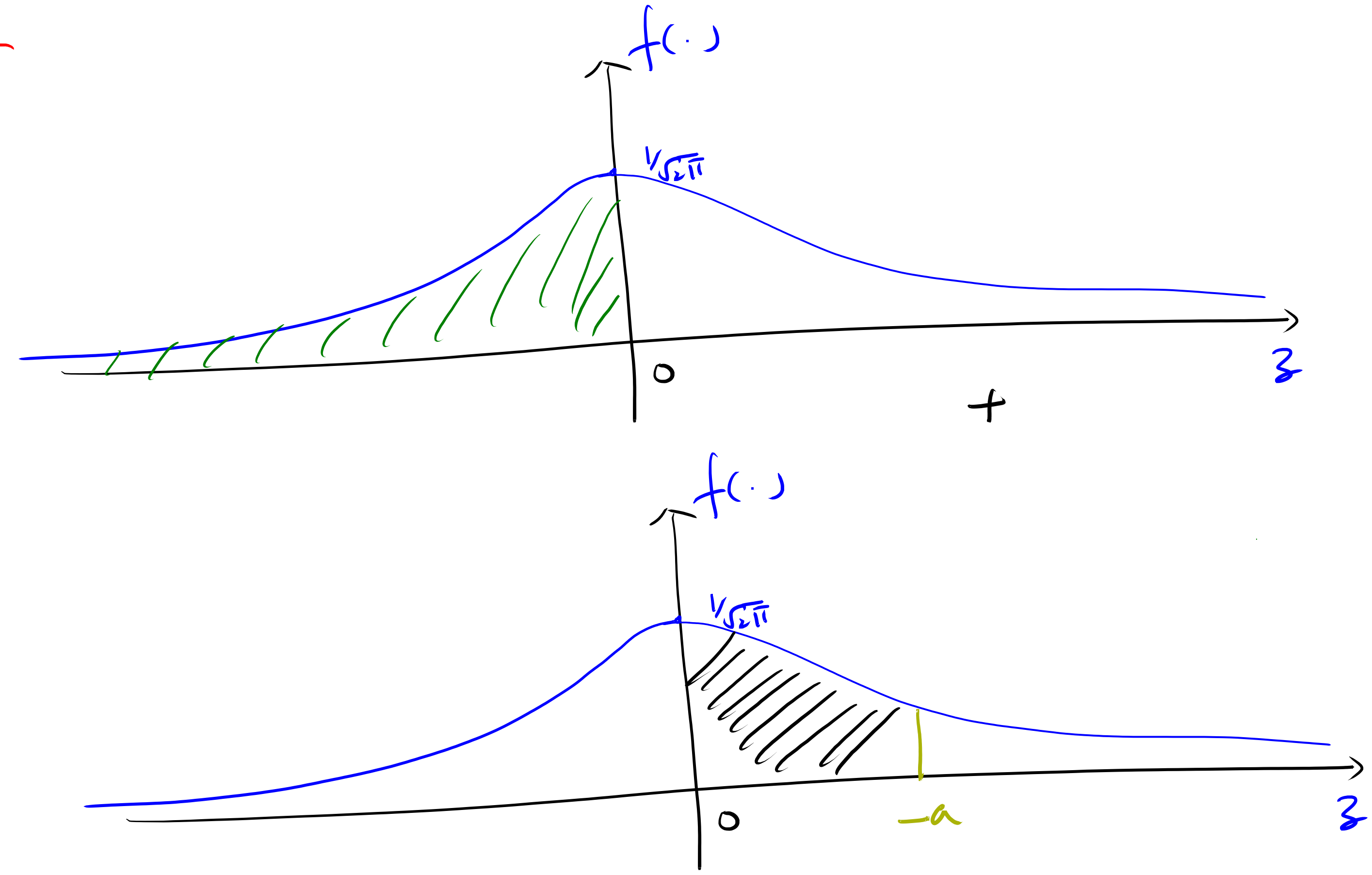
$= 1 -$ [



]

||

$= 1 -$



$$= 1 - \left[\frac{1}{2} + \int_0^{-a} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz \right]$$

$a<0$, $\displaystyle\int_{-\infty}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz = \frac{1}{2} - \underbrace{\int_0^{-a} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz}$

[It is enough to Tabulated
understand these
integrals]

2

From (1) and (2)

$$\mathbb{P}(a \le Z \le b) = \int_{-\infty}^{b} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz - \int_{-\infty}^{a} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$

Can be computed from tabulated values of

$$\int_0^{\alpha} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz \qquad \alpha > 0$$

$\square$

Definition: $X \sim \text{Normal}(\mu, \sigma^2)$ $\mu \in \mathbb{R}$, $\sigma > 0$

if X has a p.d.f given by

$$f_X(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \; ; \; x \in \mathbb{R}$$

Previously: [$Z \sim \text{Normal}(0,1)$ i.e $\mu = 0$, $\sigma = 1$]

- On Course website: – Place Normal tables of

$$\int_0^{a} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$