Recall: Poisson (λ) Distribution $\lambda > 0$

$S = \{0\} \cup \mathbb{N}$ $\quad \mathcal{F} = \mathcal{P}(S)$ $\quad \mathbb{P} : \mathcal{F} \to [0,1]$

$$k \in S - \mathbb{P}(\{k\}) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Theorem 2.2.2: $\lambda > 0$, $k \geq 1$, $n \geq \lambda$, $p = \frac{\lambda}{n}$

$A_k^{(n)} = \{k$ successes in n Bernoulli(p) trials$\}$

$$\lim_{n\to\infty} \mathbb{P}(A_k^{(n)}) = {}^nC_k \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{e^{-\lambda}\lambda^k}{k!}$$

Analysis I [fact](s):-

- $\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$

$\left\{ \begin{array}{l} e \text{ - definition} \\ e^{-\lambda} = ? \quad \lambda > 0 \\ \text{results proof} \end{array} \right.$ Ex.

$$(*) \quad e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \qquad \left[T_n = \sum_{k=1}^{n} \frac{\lambda^k}{k!} \ \& \ T_n \to e^{\lambda} \text{ as } n\to\infty \right]$$

$\{ e^{\lambda+\mu} = e^{\lambda} e^{\mu}$; results proof
Ex.

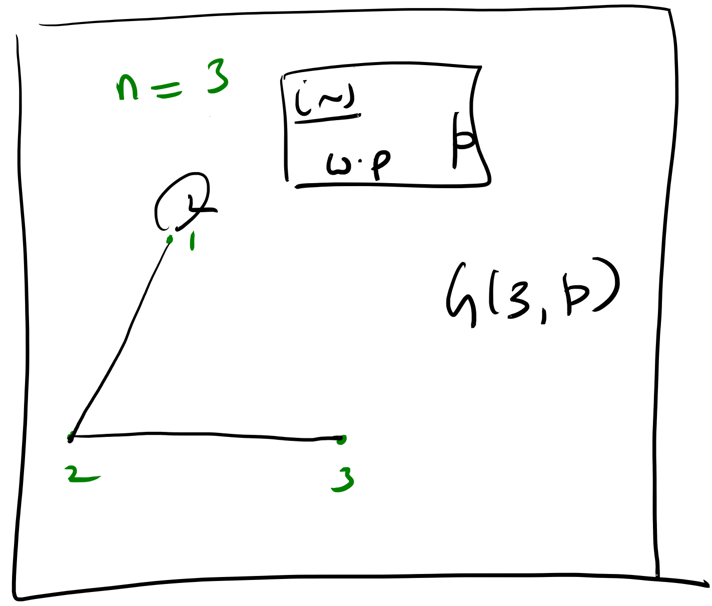
$$\underbrace{S = \{0\} \cup \mathbb{N}}_{\text{countable}} \qquad \underbrace{\mathbb{P}(\{k\}) = \frac{e^{-\lambda}\lambda^k}{k!}}_{} \ ; \quad A \subseteq S \quad \mathbb{P}(A) = \sum_{k\in A} \mathbb{P}(\{k\})$$

Axiom ② of definition

Axiom 1: $\mathbb{P}(S) = \sum_{k\in S} \mathbb{P}(\{k\}) = \sum_{k\in S} \frac{e^{-\lambda}\lambda^k}{k!}$

$$= e^{-\lambda} \sum_{k\in S} \frac{\lambda^k}{k!} \overset{(*)}{=} e^{-\lambda} e^{\lambda} = 1.$$

Ex 2.2.10 Random Graph - Erdős - Renyi (1961) - Gilbert (1959)



$V = \{1, 2, 3, \dots, n\}$ n - vertices

$E \equiv$ Edges of the graph $1 \le i \le n$, $1 \le j \le n$

[independent] $i \sim j$ w.p p

$$\boxed{G(n, p)}$$

- $d_i \equiv$ degree of vertex i

$\mathbb{P}(d_i = k) = \mathbb{P}$ (i has k-neighbours) $= \mathbb{P}$ (k success in n Bernoulli(p) trials)

$= {}^nC_k \, p^k (1-p)^{n-k}$

① — $d_i \sim$ Binomial(n, p) for all $1 \le i \le n$.

② - "$n \to \infty$" n - large $p = \frac{\lambda}{n}$ for some $\lambda > 0$.

$\Rightarrow$ [Previous result] Thm 2.2.21 $\mathbb{P}(d_i = k) = \mathbb{P}(A_k^{(n)}) \longrightarrow \frac{e^{-\lambda} \lambda^k}{k!}$

- $p = \frac{\lambda}{n}$ then $d_i^{(n)} \simeq$ Poisson (λ)

$S = \{\text{Success}, \text{Failure}\}$
$\mathbb{P}(\{\text{Success}\}) = p$ } Bernoulli (p) ;
(Two outcome)

$S = \{0, 1, \ldots, n\}$
$\mathbb{P}(\{k\}) = {}^nC_k\, p^k (1-p)^{n-k}$ } Binomial (n, p)
$k \in S$
# of successes in n-Bernoulli (p) independent trials

$S = \mathbb{N}$
$\mathbb{P}(\{k\}) = (1-p)^{k-1} p$ } Geometric (p) ;
- trial at which 1st success occurs in in Bernoulli (p) trials

$S = \{0\} \cup \mathbb{N}$
$\mathbb{P}(\{k\}) = \frac{e^{-\lambda}}{k!} \lambda^k$ } Poisson (λ)

$p = \frac{\lambda}{n}$; $\lim_{n \to \infty} \mathbb{P}$ (k successes in n independent Bernoulli (p) trials)

$S = \{1, \ldots, M\}$ $\quad \mathbb{P}(\{k\}) = \frac{1}{M}$ } uniform $\{1, \ldots, M\}$,
(Equally likely outcomes)

$\boxed{\text{Distributions encountered so far}}$

Sampling WITH and without replacement

Town has 5000 Residents, exactly 1000 of them are under the age of 18.

- $\boxed{\text{Select}}$ four people from the town at $\boxed{\text{random}}$
 Q: How many of the four are under eighteen?

I - Sample with replacement : Each selection could be any of the 5000 members (equally likely) ; and is independent of other selections.

Ans:- Bernoulli $(\frac{1}{5})$ trials $= 4$

techniques developed so far will give answer.

$$\mathbb{P}(k \text{ out of four under eighteen}) = {}^4C_k \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{4-k}$$

$k = 0, 1, 2, 3, 4.$

I – Peculiarities :- select four people – you may have chosen person two times / multiple times.

II – Sampling WITHOUT Replacement : Assume that if an individual is chosen in a selection then they are NOT available to be picked in a later selection. Ans :- We can consider all possible groups of 4 in the population ; choose a group uniformly from them (i.e each group is equally likely to be chosen) :- NOT independent Bernoulli trials.

Q :- What is that Probability that in the group of 4 that exactly two are under the age of eighteen?

A :- $\binom{5000}{4}$ – ways of selecting 4 residents from the town of 5000.

1000 of them are under age of eighteen. – select exactly 2 from this section — $\binom{1000}{2}$

4000 of them who are NOT under age of eighteen. – select exactly 2 from this section — $\binom{4000}{2}$

$\therefore$ Probability of choosing exactly two residents under the age of eighteen $= \dfrac{\binom{1000}{2}\binom{4000}{2}}{\binom{5000}{4}}$

Compare

$P(\{$exactly two out of 4 to be under age of 18$\})$

Sample with replacement

Binomial$\left(4, \frac{1}{5}\right)$

$^4C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^2$

≈ 0.1536

Sample without replacement

$$\frac{\binom{1000}{2}\binom{4000}{2}}{\binom{5000}{4}}$$

≈ 0.153592

answers are not equal but they are "close".

Statistics application:- We will show if sample chosen is small relative to size of the population, then the two sampling methods will give similar results.

2.3.1 Hypergeometric Distributions

N – Population size [Balls] e.g.

r – out of N have a certain characteristic [r – red Balls]

m – sample chosen from N WITHOUT replacement.

Q:– How many red balls | population with a certain characteristic are chosen?

A :– largest possible $\equiv \min\{m, r\}$

minimum possible $\equiv \max\{0, m - (N - r)\}$

$S = \max\{0, m-(N-r)\}$ to $\min\{m, r\}$

$$\mathbb{P}(\{k\}) = \frac{\binom{r}{k}\binom{N-r}{m-k}}{\binom{N}{m}}$$

Hypergeometric (N, r, m)