Recall: (Continuous) Probability spaces

Eg $S = (0,1)$ – uncountable
– choosing a # uniformly from $(0,1)$

– $\mathbb{P}(\{x\}) = 0 \quad \forall x \in S$

(Wanted): – $\mathbb{P}([3/4, 1]) = 1/4$ (length of interval)

– $A \subseteq (0,1) \equiv$ length of the set A ?

Definition 5.1.1: S – sample space ; σ-field on $S \equiv \mathcal{F}$

① $S \in \mathcal{F}$ ② $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.

③ $A_1, A_2, \ldots$ are countable collection in $\mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

– [contains whole space, ① closed < under complementation ②, under countable unions ③]

an element of σ-field $\equiv$ called event

Definition 5.1.2: S be a sample space, $\mathcal{F}$ – σ-field on S

$\mathbb{P} : \mathcal{F} \to [0,1]$

① $\mathbb{P}(S) = 1$ &

② $\{E_k\}_{k=1}^{\infty}$ $E_l \cap E_j = \phi$ $l \neq j$

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

Example 5.1.3:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

$S = \mathbb{R}$ $\quad$ $\mathcal{F}$ – (Borel σ-field) – smallest σ-field that contains all intervals.

$A \in \mathcal{F}$ $\equiv$ " think of it as an interval "

$$\mathbb{P}(A) = \int_A f(x)\,dx$$

- $\mathbb{P}(\{a\}) = 0 \quad \forall\ a \in (0,1)$
- $\mathbb{P}([\alpha,\beta)) = \beta - \alpha \qquad (\alpha,\beta) \subseteq (0,1)$
- $(c,d) \cap (0,1) = \phi$

$$\mathbb{P}((c,d)) = \int_c^d f(x)\,dx = \int_c^d 0\,dx = 0$$

(*) $\mathbb{P}((0,1)) = \int_0^1 f(x)\,dx = \int_0^1 1\,dx = x\Big|_0^1 = 1$

Definition 5.1.4 : Let $f : \mathbb{R} \to \mathbb{R}$ is called a density function if f satisfies the following

1. $f(x) \geq 0$

2. f is piecewise continuous

3. $$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

Theorem 5.1.5 Let $f(\cdot)$ be a density function. Define

$$\mathbb{P}(A) = \int_A f(x)\, dx \quad \text{for any event } A \subseteq \mathbb{R}$$

Then $\mathbb{P}$ defines a probability on $\mathbb{R}$ & f is called the density function for the probability $\mathbb{P}$.