Recall:-

- $S$ - sample space  $\mathcal{F}$ - events $\equiv \mathcal{P}(S)$ Temporary Definition
  $\mathbb{P}$ - Probability on $S$ - Countable [finite]
- $X : S \to T$  Discrete Random variables.
  p.m.f · $f_X(t) = \mathbb{P}(X=t) \quad \forall t \in T$ [Distribution of $X$]
- Outcome spaces $\equiv$ Countable [infinite / finite]
  e.g $\mathbb{N}$ , $\{1,2,3,4,5,6\}$

Question 1:- How to choose a number randomly [uniform] from the set $S=(0,1)$ ?

Difference:- - $S=(0,1)$ is not Countable
Q: Can we assign Probabilities to elements of $S$ to answer question 1 ?

- This is very much possible.
- Technical difficulties to overcome.
  - Develop / define - precise setup

## Key Technical obstacles for Question 1 or $\mathfrak{Q}$

- In Discrete setting, "Uniform" meant every outcome in the sample space $S$ was equally likely

literal translate $\Rightarrow$ $S = (0,1)$ & every outcome is equally likely.

So, let $0 \le p \le 1$ & $\mathbb{P}(\{x\}) = p \quad \forall x \in S$

Why this Fails :- $E = \{ \frac{1}{n} : n \geq 2 \} = \{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots \}$

- $\mathbb{P}(E) = \sum_{k=2}^{\infty} \mathbb{P}(\{1/k\})$

$= p + p + p + \cdots$

If the series on r.H.S. converged then $p = 0$.

[ $p > 0$ cannot happen ]

- $S$ - sample space, $S = (0,1)$. $\mathbb{P}(S) = 1$ !

then: $\mathbb{P}(S) = \mathbb{P}(\bigcup_{x \in (0,1)} \{x\})$

Countable sequence of events: $\{E_k\}_{k=1}^{\infty}$ are disjoint, $\mathbb{P}(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$

not necessarily valid

uncountable $\bigcup_{x \in S} \{x\}$

$= \sum_{x \in (0,1)} \mathbb{P}(\{x\})$

$= \sum_{x \in (0,1)} 0 = 0$ !

Contradiction

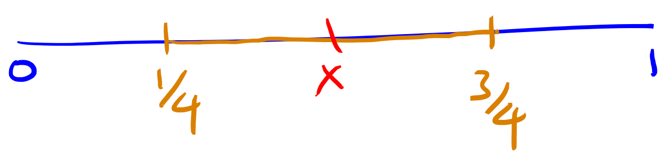
Obstacle: $\mathbb{P}(\{x\}) = 0 \quad \forall x \in S$ $\dashrightarrow$ Compute? $\mathbb{P}([\frac{1}{4}, \frac{3}{4}]) = ?$

- Cannot sum over probabilities of each outcome in $[\frac{1}{4}, \frac{3}{4}]$ as the set is UNCOUNTABLE

Key step in Resolution: Interpret correctly what we mean by "uniform" in this situation when $S = (0,1)$

- tempting / correct: $[\frac{1}{4}, \frac{3}{4}]$ – interval / event of interest

would like to define a probability on $S$ such that




$\mathbb{P}([\frac{1}{4}, \frac{3}{4}]) = \frac{1}{2}$ as $[\frac{1}{4}, \frac{3}{4}]$ – is $\frac{1}{2}$ length of $(0,1)$

Option 1 $\equiv$ lets do the following: define $\mathbb{P}(A) = $ length of $A$, $A \subseteq S = (0,1)$

✓ no contradiction to earlier calculation:
- $\mathbb{P}(\{x\}) = \mathbb{P}([x,x]) = 0$
- $\mathbb{P}(S) = \mathbb{P}((0,1)) = 1$

Unfortunate: (not in the scope of this class) – $\{x\} \Rightarrow$ length $= 0$

$a, b \in (0,1)$: $\left.\begin{array}{l}(a,b) \\ (a,b] \\ [a,b] \\ [a,b)\end{array}\right\} \Rightarrow$ length $= b-a$

$\boxed{\text{OBSTACLE}}$

It is not possible to define lengths for any $A \subseteq (0,1)$.

Resolution :- • Reconsider what we mean by events

Event $A$ : $\Leftarrow$ $A \subseteq S$ [ Not going to work ]

• $S = (0,1)$ we want to choose a number uniformly in $(0,1)$ — At the very least :- need event to include

• intervals ; Axiom of Probability → - Unions [Countable] } of intervals
- Complements
- Combinations

---

**Definition 5.1.1** [ $\sigma$- field ] $S$ - a sample space
$\mathcal{F}$ - $\sigma$-field on $S$ is a collection of subsets of $S$ such that

1. $S \in \mathcal{F}$
2. $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
3. $A_1, A_2, \ldots A_n, \ldots$ a countable collection of sets in $\mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

We shall refer to an element of the $\sigma$-field $\mathcal{F}$ as an event.

- $S = (0,1)$ — Designate — Smallest $\sigma$-field that contain all intervals — Borel $\sigma$-field $\equiv \mathcal{F}$

Events space

$A \in \mathcal{F}$ — One can define length of $A$

Scope of this class

- $S = \mathbb{R}$ — or $S \subseteq \mathbb{R}$ — Then also we will take / Designate event space as Borel $\sigma$-field $\equiv \mathcal{F}$.

Definition 5.1.2 :- [Probability Space Axioms] Let $S$ be a sample $\mathcal{F}$ be a $\sigma$-field on $S$. A "probability" is a function

$\mathbb{P} : \mathcal{F} \to [0,1]$ such that

(1) $\mathbb{P}(S) = 1$

(2) $E_1, E_2, \ldots E_n, \ldots$ are countable collection of disjoint elements in $\mathcal{F}$ then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

$(S, \mathcal{F}, \mathbb{P})$ — referred to as Probability space.

Remarks: • – no deviation for our previous set up

$S$ – Countable $\qquad \mathcal{P}(S) \equiv$ that is a $\sigma$-field.

• – $S = (a,b)$ or $S = \mathbb{R}$ – Borel $\sigma$-field as event space $= \mathcal{F}$

Smallest collection that contain intervals

1 $S \in \mathcal{F}$

2 $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

3 $A_1 \ldots A_n \ldots \in \mathcal{F}$ c'tble $\quad \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

• Going forward we'll do the following:

restricts – $S = \mathbb{R}$ or $\boxed{\text{subsets of } \mathbb{R}}$

– $\underbrace{A \in \mathcal{F} \text{ event}}_{\text{Prove Something}}$ – Being interval

– finite / c'tble intervals

– complements

Only consider these union of

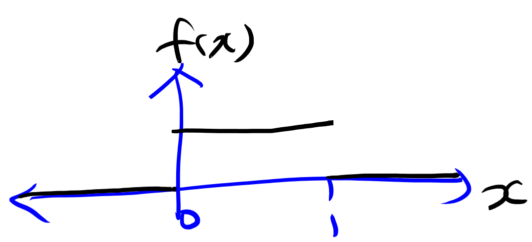
## 5.1.1. Probability Densities on $\mathbb{R}$

– Primary method of defining Probabilities on $\mathbb{R}$ is via Density function.

**Example 5.1.5.** ( Choosing a number uniformly in $[0,1]$ )

$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$



let $A$ be event on $\mathbb{R}$ : define

$$\mathbb{P}(A) = \int_A f(x)\,dx \quad -(*)$$

E.g. . $A = [\frac{3}{4}, 1] \Rightarrow \mathbb{P}([3/4, 1]) = \int_{3/4}^{1} f(x)\,dx$

$$\Rightarrow \mathbb{P}([3/4, 1]) = \int_{3/4}^{1} 1\,dx = x\Big|_{3/4}^{1}$$

$$= 1 - 3/4 = 1/4$$

. $A = [a, b]; 0 < a, b < 1$

$$\mathbb{P}([a, b]) = \int_a^b f(x)\,dx = \int_a^b 1\,dx$$

$$= x\Big|_a^b = b - a$$

. $\mathbb{P}$( interval in $(0,1)$) = length of the interval in $(0,1)$.

<u>Show next time</u>

. $\mathbb{P}$ defined by $(*)$ is indeed a Probability

- $\mathbb{P}(\{c\}) = 0$
- $[a,b] \subseteq (0,1)^c$ then $\mathbb{P}((a\ b)) = 0$

$\Rightarrow$ $\mathbb{P}$ is model for choosing a number uniformly in $(0,1)$