**Recall**

$(S, \mathcal{F}, \mathbb{P})$ — $X : S \to T$ with $T$ — countable (Discrete random variable)

Assumed $S$ - countable till now.

- Distribution of $X$ — $\text{Range}(X) = T$

p.m.f. : $f_X(t) = \mathbb{P}(X = t)$ $\quad f_X : T \to [0,1]$

- $B \subseteq T$ $\quad \mathcal{P}(B) = \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B))$

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**$X, Y$ are two discrete random variables** : $X : S \to U \quad Y : S \to T$

- independent $\quad \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$

Equivalent $\quad \mathbb{P}(X = u, Y = t) = \mathbb{P}(X = u)\, \mathbb{P}(Y = t)$ $\quad \forall u \in U, \; t \in T.$

- Conditional distributions $\quad$ E.g. $A = \{X = x\}$ for $x \in S$

$$Y \mid X = x \;\equiv\; Y \mid A \text{ - Conditional}$$

distribution of $Y$ given the event $A$ ; $\mathbb{P}(A) > 0$

$$B \subseteq T, \quad \mathcal{P}(B) = \mathbb{P}(Y \in B \mid A) = \frac{\mathbb{P}(Y \in B, A)}{\mathbb{P}(A)}$$

- Joint distribution of $X$ and $Y$ : $U \times T$ — Probability on it

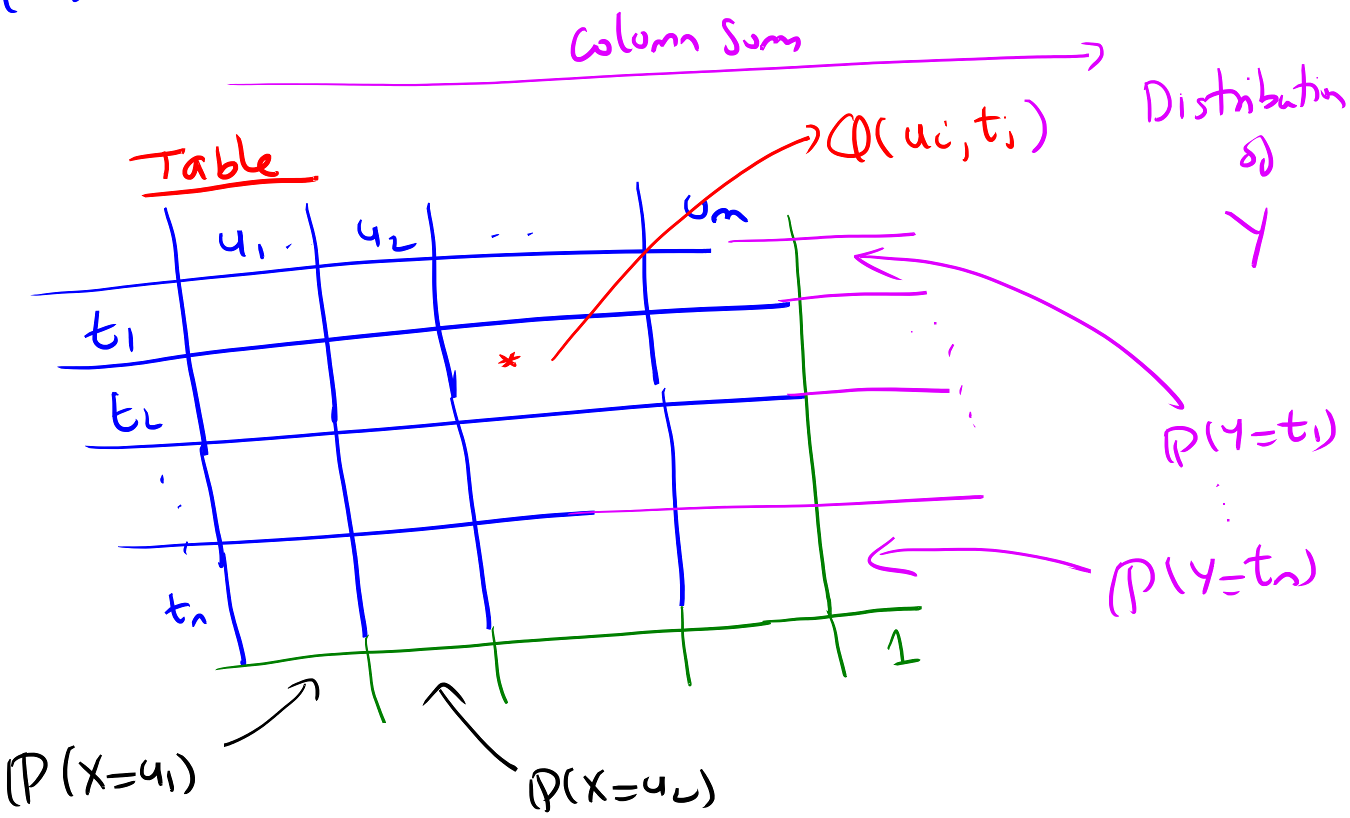
$$\mathcal{P}(\{(u,t)\}) = \mathbb{P}(X = u, Y = t) \quad \forall u \in U \; \forall t \in T$$

$U, T$ — are finite

$|U| = m$

$|T| = n$

Table




Row Sums

Distribution of $X$

independent $\iff$ $\mathcal{P}(u_i, t_j) = \mathbb{P}(X = u_i) \times \mathbb{P}(Y = t_j)$

## 3.2.3. Memoryless Property of Geometric Random variable

**Intuitive** Toss a coin 40 times & No Head appears

$\underbrace{T \; T \cdots\cdots}_{40 \text{ tosses}} T$

Toss the coin 60 times & Head appears in 65th toss for the 1st time

$\underbrace{T \; T \cdots\cdots}_{40 \text{ tosses}} T, \underbrace{T \; T \; T \; T}_{4} H \cdots\cdots \uparrow$ 100th toss

$\mathbb{P}$( first head appears in 45th toss | No head appears in the 1st 40 tosses )

$\overset{\text{intuitively}}{=}$ $\mathbb{P}$( ....., TTTT H | No head appears in the 1st 40 tosses )

Independence

.....

Absence of memory

$=$ $\mathbb{P}$( head occurs for the 1st time in 5 toss of a coin )

**Example 3.2.11**: Suppose we toss a fair coin until the 1st head appears.

$X \equiv$ trial at which 1st head occurs

$X \sim \text{Geometric}(1/2)$ $-$ $\mathbb{P}(X=k) = \left(\frac{1}{2}\right)^{k-1} \frac{1}{2} = \frac{1}{2^k}$

$\forall k \geq 1$

$m \geq 1$ :  $\mathbb{P}(X > m) = \mathbb{P}\left(\bigcup_{k=m+1}^{\infty} X=k\right) = \underbrace{\sum_{k=m+1}^{\infty} \mathbb{P}(X=k) = \cdots = \frac{1}{2^m}}$

$n \geq 1$

$$\mathbb{P}(X \;\; m+n) = \text{imitate} \cdots = \frac{1}{2^{m+n}}$$

$$\mathbb{P}(X > m+n \mid X > m)$$

$$= \frac{\mathbb{P}(X > m+n \,,\, X > m)}{\mathbb{P}(X > m)}$$

$\{X > m+n\} \subseteq \{X > m\}$

$$= \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > m)}$$

$$= \frac{\frac{1}{2^{m+n}}}{1/2^m}$$

$$= \frac{1}{2^n} = \mathbb{P}(X > n)$$

Remark: True if $X \sim \text{Geometric}(p)$

$$\mathbb{P}(\underbrace{T \cdots T}_{m \text{ toss}}, \underbrace{T \cdots T}_{n \text{ toss}}, \cdots \mid \underbrace{T\,T \cdots T}_{m \text{ tosses}} \cdots\cdots) = \mathbb{P}(\underbrace{T \cdots T}_{n \text{ toss}}, \cdots)$$

memoryless property of Geometric distribution

– Conditional on the fact that there are no heads in the 1st m tosses of a coin then the probability of no head in n further tosses of the coin is the same as probability of no heads in n tosses of the same coin.

## 3.2.4 Multinomial Distributions : E.g $X \sim$ Uniform $\{1,2\}$

$Y$ – # of heads in $X$ tosses

– $(X,Y)$ – Joint distribution

## Example 3.2.12 [ $n \geq 1$, $\boxed{(X_1, \ldots, X_n)}$ ] $\boxed{(Y_1, \ldots, Y_k)}$

- $n$ – independent and identical trials of an experiment with $k$– outcomes, for $k > 1$.

- let $X_i$ – represent the outcome of trial $i$ $\quad T = \{1, 2, \ldots k\}$

Assume $\quad \mathbb{P}(X_i = j) = P_j \quad j \in T$

$(0 \leq p_j \leq 1, \quad \sum_{j=1}^{k} p_j = 1)$

$X_i$ are independent & have same distribution

- let $Y_j$ represent the # of the $n$ trials that result in the $j^{th}$ outcome, $1 \leq j \leq k$

Note :- $\displaystyle \sum_{j=1}^{k} Y_j = n$

$\{Y_j\}$ are Dependent random variables

Easy calculation:- $x_i \in T \quad 1 \le i \le l$

$-B = \{ X_1 = x_1, \ldots, X_l = x_l \}, \; 1 \le l \le n$

$$\mathbb{P}(B) = \; ? = \mathbb{P}(X_1 = x_1, \ldots, X_l = x_l) \overset{\text{independence}}{=} \prod_{i=1}^{l} \mathbb{P}(X_i = x_i)$$

Harder Calculation

$$C = \{ Y_1 = y_1, \ldots \quad Y_k = y_k \} \quad ; \quad \sum_{j=1}^{k} y_j = n \; ; \quad 0 \le y_i \le n$$

$$\mathbb{P}(C) = \; ?$$

$\omega \in C$

$(\omega_1, \ldots, \omega_n)$

$$\mathbb{P}(\{\omega\}) = \mathbb{P}\begin{pmatrix} \text{outcome } 1 \text{ occurs in } y_1 \text{ trials} \\ \vdots \\ \text{outcome } j \text{ occurs in } y_j \text{ trials} \\ \vdots \\ \text{outcome } k \text{ occurs in } y_k \text{ trials} \end{pmatrix}$$

n trials are independent

$$= p_1^{y_1} \cdots p_j^{y_j} \cdots p_k^{y_k}$$

$$= \prod_{j=1}^{k} p_j^{y_j}$$

$$\mathbb{P}(C) = \sum_{\omega \in C} \mathbb{P}(\{\omega\}) = \sum_{\omega \in C} \prod_{j=1}^{k} p_j^{y_j}$$

$$= |C| \prod_{j=1}^{k} p_j^{y_j} \quad ; \quad \text{if } \sum_{j=1}^{k} y_j = n$$

Ex:

n balls $- \{ y_j : 1 \le j \le k \}$ $\quad \sum_{j=1}^{k} y_j = n$

$$|C| = \underset{1}{\boxed{y_1}} \cdots\cdots \underset{k}{\boxed{y_k}} = \frac{n!}{y_1! \cdots y_k!}$$

Boxes

$$\mathbb{P}(Y_1=y_1, \ldots, Y_k=y_k) = \begin{cases} \dfrac{n!}{y_1! \cdots y_k!} \prod_{j=1}^{k} p_j^{y_j} ; & \sum_{j=1}^{k} y_j = n ; \ 0 \le y_i \\ 0 & \text{otherwise} \end{cases}$$

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Conventions :- $(S, \mathcal{F}, \mathbb{P})$ $X : S \to T$ $T \subseteq \mathbb{R}$ $T$ - Countable

A discrete random $X$ is called a "constant" (or constant random variable) if $\exists\ c \in \mathbb{R}$ such that

$$\mathbb{P}(X=c) = 1. \quad \text{ie } T = \{c\}$$

• The constant random variable $X$ is independent of itself.

To show : $\mathbb{P}(X \in A, X \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(X \in B)$
$A, B \subseteq T$

$A = \{c\} = B$ $\quad \mathbb{P}(X=c, X=c) = \mathbb{P}(X=c)\,\mathbb{P}(X=c)$

ie $\quad \mathbb{P}(X=c) = (\mathbb{P}(X=c))^2$

As $\quad \mathbb{P}(X=c) = 1$ This is true.

• If $X$ is discrete random variable & $X$ is independent of itself then $X$ is a constant.

Ex - To complete