Recall :-

$(S, \mathcal{F}, \mathbb{P})$ - S - countable, $\mathcal{F} = \mathcal{P}(S)$

$X : S \to T$ — T - countable — Discrete Random variables

need not be countable

- Distribution of X :- $Q(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(X \in B)$

- Probability mass function :- p.m.f.

$f_X : T \to [0,1]$; $f_X(t) = \mathbb{P}(X = t)$ $\forall t \in T$ — Distribution of X.

- X, Y are two random variables - they have the same distribution if $f_X(\cdot) \equiv f_Y(\cdot)$

ie $\text{Range}(X) = \text{Range}(Y)$ & $f_X \equiv f_Y$

- X, Y are independent random variables if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$$

ie it is enough to check : $\mathbb{P}(X = s, Y = t) = \mathbb{P}(X = s)\mathbb{P}(Y = t)$

$\forall s \in \text{Range}(X)$ and $\forall t \in \text{Range}(Y)$

- $X_1, X_2, \ldots, X_n$ are mutually independent if $(X_j \in A_j)_{j=1,\ldots,n}$ are mutually independent for all events A_j in the Range (X_j)..

$\{X_t\}_{t \in I}$, I is any index set ; is mutually independent if every finite sub-collection is mutually independent.

Example 3.2.4 :- $\{X_i : 1 \le i \le n\}$ $X_i \sim \text{Geometric}(p)$

What is the probability that all of these random variables are larger than some positive integer N?

Ans :

- $X_1 \sim \text{Geometric}(p)$

$$\mathbb{P}(X_1 > N) \overset{X_1 \sim \text{Geometric}(p)}{=} (1-p)^N$$

- $$\mathbb{P}(X_1 > N, X_2 > N, \ldots, X_n > N) \overset{\text{independence}}{=} \prod_{i=1}^{n} \mathbb{P}(X_i > N)$$

$$= (\mathbb{P}(X_1 > N))^n$$

$$= (1-p)^{nN}$$

$\square$

3.2.2 Conditional, Joint, & Marginal distribution

Example 3.2.6: $\cdot$ $X \equiv$ a number uniformly chosen from $\{1,2\}$

$Y \equiv$ be the number of heads in X tosses of a fair coin. [Intuitively its clear that X and Y are dependent]

- $X=1$ then Range$(Y) = \{0,1\}$
- $X=2$ then Range$(Y) = \{0,1,2\}$

Distribution of X:

$X \sim$ Uniform$\{1,2\}$ $\longrightarrow$ Range$(X) = \{1,2\}$, $\mathbb{P}(X=1) = \mathbb{P}(X=2) = \frac{1}{2}$

$\boxed{X=1}$ –

$$\mathbb{P}(Y=0 \mid X=1) = \frac{1}{2}$$

$$\mathbb{P}(Y=1 \mid X=1) = \frac{1}{2}$$

$\boxed{X=2}$

$$\mathbb{P}(Y=0 \mid X=2) = {}^2C_0\left(\tfrac{1}{2}\right)^2 = \frac{1}{4}$$

$$\mathbb{P}(Y=1 \mid X=2) = {}^2C_1\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \frac{1}{2}$$

$$\mathbb{P}(Y=2 \mid X=2) = {}^2C_2\left(\tfrac{1}{2}\right)^2 = \frac{1}{4}$$

Definition 3.2.5:- let Y be a random variable on a sample space S. let $A \subseteq S$ be an event such that $\mathbb{P}(A) > 0$. Then the Probability $\mathbb{Q}$ described by

$$\mathbb{Q}(B) = \mathbb{P}(Y \in B \mid A) = \frac{\mathbb{P}((Y\in B) \cap A)}{\mathbb{P}(A)}$$

is called the "Conditional distribution" of Y given the event A.

Ex:- $\mathbb{Q}$ is indeed a Probability on S.

Example 3.26 (Contd)

$A = \{X=1\}$ and we showed

$$\mathbb{P}(Y=0 \mid A) = \mathbb{P}(Y=0 \mid X=1) = \frac{1}{2}$$

$$\mathbb{P}(Y=1 \mid A) = \mathbb{P}(Y=1 \mid X=1) = \frac{1}{2}$$

$Y \mid X=1 \sim \text{Bernoulli}(1/2)$

– Conditional distribution of Y given $X=1$ is Bernoulli $(1/2)$

$A = \{X=2\}$

$$\mathbb{P}(Y=0 \mid A) = \mathbb{P}(Y=0 \mid X=2) = {}^2C_0 \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

$$\mathbb{P}(Y=1 \mid A) = \mathbb{P}(Y=1 \mid X=2) = {}^2C_1 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$\mathbb{P}(Y=2 \mid A) = \mathbb{P}(Y=2 \mid X=2) = {}^2C_2 \left(\tfrac{1}{2}\right)^2 = 1/4$$

$Y \mid X=2 \sim \text{Binomial}(2, \tfrac{1}{2})$

– Conditional distribution of Y given $X=2$ is Binomial $(2, \tfrac{1}{2})$

$X \sim \text{uniform}\{1,2\}$; $Y \mid X=x \sim \text{Binomial}(x, \tfrac{1}{2})$

$x \in \{1,2\}$

Definition 3.27: If X and Y are discrete random variables, the "joint distribution" of X and Y is the probability $\mathbb{Q}$ on the pair of values in the ranges of X and Y defined by

$$\mathbb{Q}((a,b)) = \mathbb{P}(X=a, Y=b) \quad \forall a \in \text{Range}(X),\ b \in \text{Range}(Y)$$

Ex:-

$X: S \to T$ T - Countable

$Y: S \to U$ U - Countable $\Rightarrow T \times U$ is countable , $\mathcal{G} = \mathcal{P}(T \times U)$

$$\mathbb{Q} : \mathcal{G} \to [0,1]$$

$$\mathbb{Q}((a,b)) = \mathbb{P}(X=a, Y=b)$$

is indeed a Probability on $T \times U$

Example 3.2.6 (contd.)

$X \sim \text{uniform}\{1,2\}$

$Y \mid X = x \sim \text{Binomial}(x, \tfrac{1}{2})$

$Y: S \to \{0,1,2\}$

$X: S \to \{1,2\}$

Joint distribution of X and Y

$$\left.\begin{aligned}
\mathbb{P}(X=1, Y=0) &= \mathbb{P}(Y=0 \mid X=1)\,\mathbb{P}(X=1) = \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}\\
\mathbb{P}(X=1, Y=1) &= \mathbb{P}(Y=1 \mid X=1)\,\mathbb{P}(X=1) = \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}\\
\mathbb{P}(X=1, Y=2) &= \mathbb{P}(Y=2 \mid X=1)\,\mathbb{P}(X=1) = 0 \cdot \tfrac{1}{2} = 0\\
\mathbb{P}(X=2, Y=0) &= \mathbb{P}(Y=0 \mid X=2)\,\mathbb{P}(X=2) = \tfrac{1}{4}\cdot\tfrac{1}{2} = \tfrac{1}{8}\\
\mathbb{P}(X=2, Y=1) &= \mathbb{P}(Y=1 \mid X=2)\,\mathbb{P}(X=2) = \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}\\
\mathbb{P}(X=2, Y=2) &= \mathbb{P}(Y=2 \mid X=2)\,\mathbb{P}(X=2) = \tfrac{1}{4}\cdot\tfrac{1}{2} = \tfrac{1}{8}
\end{aligned}\right\}\ \varphi \text{ Joint dist. of } X \,\&\, Y$$

$\varphi(a,b) = \mathbb{P}(X=a, Y=b)$	$X=1$	$X=2$
$Y=0$	$1/4$	$1/8$
$Y=1$	$1/4$	$1/4$
$Y=2$	0	$1/8$

– Complete picture / Relationship between X and Y.

- $X \sim \text{Uniform}\{1,2\}$; $Y \mid X=x \sim \text{Binomial}(x, \tfrac{1}{2})$ $x \in \{1,2\}$

- Joint Distribution of X & Y

	$X=1$	$X=2$
$Y=0$	$1/4$	$1/8$
$Y=1$	$1/4$	$1/4$
$Y=2$	0	$1/8$

ie Given $Y=0$ what is the conditional distribution of X?

$$\mathbb{P}(X=1 \mid Y=0) = \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0)} = \frac{\mathbb{P}(Y=0 \mid X=1)\,\mathbb{P}(X=1)}{\mathbb{P}(Y=0 \mid X=1)\,\mathbb{P}(X=1) + \mathbb{P}(Y=0 \mid X=2)\,\mathbb{P}(X=2)}$$

Bayes Theorem

$$= \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0, X=1 \cup Y=0, X=2)}$$

$$= \frac{\mathbb{P}(X=1, Y=0)}{\mathbb{P}(Y=0, X=1) + \mathbb{P}(Y=0, X=2)}$$

$$= \frac{1/4}{1/4 + 1/8} = \frac{1/4}{3/8} = \frac{2}{3}$$

$$\mathbb{P}(X=2 \mid Y=0) = \frac{\mathbb{P}(X=2, Y=0)}{\mathbb{P}(Y=0)} = \frac{\mathbb{P}(X=2, Y=0)}{\mathbb{P}(X=1, Y=0) + \mathbb{P}(X=2, Y=0)}$$

$$= \frac{1/8}{1/4 + 1/8} = \frac{1/8}{3/8} = \frac{1}{3}$$

$X \mid Y=0$ = Conditional distribution of X given $Y=0$. $\left\{ \mathbb{P}(X=1 \mid Y=0) = 2/3 \right.$ & $\mathbb{P}(X=2 \mid Y=0) = \frac{1}{3}$

Similarly one can compute $X \mid Y=1$, $X \mid Y=2$
Conditional distribution

- Joint Distribution of X & Y

$\mathbb{P}(X=x, Y=y)$	$X=1$	$X=2$	
$Y=0$	$1/4$	$1/8$	$\leftarrow 3/8 = \mathbb{P}(Y=0)$
$Y=1$	$1/4$	$1/4$	$\leftarrow 1/2 = \mathbb{P}(Y=1)$
$Y=2$	0	$1/8$	$\leftarrow 1/8 = \mathbb{P}(Y=2)$
	$1/2$	$1/2$	

Column sum of the Joint distribution table

$\text{Range}(Y) = \{0,1,2\}$ & $\mathbb{P}(Y=0) = 3/8$, $\mathbb{P}(Y=1) = \frac{1}{2}$, $\mathbb{P}(Y=2) = 1/8$

- Distribution of Y. - Marginal distribution of Y

Row sum of the Joint distribution table

$$\mathbb{P}(X=1) = \mathbb{P}(X=1, Y=0 \cup X=1, Y=1 \cup X=1, Y=2) = \mathbb{P}(X=1, Y=0) + \mathbb{P}(X=1, Y=1) + \mathbb{P}(X=1, Y=2) = \frac{1}{2}$$

Illarly,
$$\mathbb{P}(X=2) = \mathbb{P}(X=2, Y=0) + \cdots + \mathbb{P}(X=2, Y=2) = \frac{1}{2}$$

$\text{Range}(X) = \{1,2\}$ — Distribution of X - Marginal distribution of X.

Calculation

Marginal of Y: $\mathbb{P}(Y=0) = 3/8$

Marginal of X: $\mathbb{P}(X=1) = \frac{1}{2}$

Joint: $\mathbb{P}(Y=0, X=1) = 1/4$

$$\mathbb{P}(Y=0, X=1) \neq \mathbb{P}(Y=0)\,\mathbb{P}(X=1)$$

$\Rightarrow$ X and Y are NOT independent.

$X : S \to T$; $Y : S \to \mathcal{U}$

- Conditional distribution of $Y \mid A$ A any event

$Y \mid X = x \equiv$ conditional distribution of Y given $X = x$

- Joint distribution of (X, Y)

$$\varphi(a,b) = \mathbb{P}(X = a, Y = b) \qquad a \in \text{Range}(X), \; b \in \text{Range}(Y)$$

- Marginal distribution of X and Y

$$f_X(a) = \mathbb{P}(X = a), \; a \in T \quad \& \quad f_Y(b) = \mathbb{P}(Y = b), \; b \in \mathcal{U}.$$