Recall:- Discrete Random variables

- $(S, \mathcal{F}, \mathbb{P})$ — S: Countable or finite

$X: S \longrightarrow T$ — Discrete random variable

any function

typical $T \subseteq \mathbb{R}$ - not necessary

- Distribution of X $(T, \mathcal{F}_T, \varphi)$

$$\varphi(B) = \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}(B))$$

Example 3.1.4 - A certain lottery - 3 digit number that is randomly selected from (000 to 999).

- If a ticket matches then winner get 200 Rs.

If a ticket matches exactly 2 of out of 3 digits then winner get 20 Rs

otherwise — get nothing.

Let X be the earnings on a ticket.

$X: S \to \{0, 20, 200\} := T$

$S = \{000, 001, \dots, 999\}$

Range(X) $\subseteq \mathbb{R}$

$$\varphi(\{200\}) = \mathbb{P}(X = 200) = \frac{1}{1000}$$

$$\varphi(\{20\}) = \mathbb{P}(X = 20) = \frac{\binom{3}{2} \cdot 9}{1000} = \frac{27}{1000}$$

$$\varphi(B) = \sum_{\omega \in B} \varphi(\{\omega\})$$

$$\varphi(\{0\}) = \mathbb{P}(X = 0) = 1 - \varphi(\{200\}) - \varphi(\{20\}) = \frac{972}{1000}$$

$\square$

Definition 3.1.5 :- A "discrete random variable" is a function

$X : S \to T$ where S - countable (includes finite)
& T - countable (includes finite) $\subseteq \mathbb{R}$

S is equipped with a Probability $\mathbb{P}$.

From Theorem 3.1.2 $\mathbb{P}$ & X generate a Probability on T & since T is countable (i.e. discrete); the distribution of X is governed by : $f_X : T \to [0,1]$ where

$$f_X(t) = \mathbb{P}(X = t) \quad \&$$

$f_X(\cdot)$ is referred to as the probability mass function of X.

Indeed : $P(X \in A) = \varphi(A) = \sum_{t \in A} \varphi(\{t\}) = \sum_{t \in A} \mathbb{P}(X = t)$

$$= \sum_{t \in A} f_X(t)$$

Keys :-

- $X : S \to T$ discrete random variable
  Range$(X) \subseteq \mathbb{R}$, T - countable

  Probability mass function $\left\{ \begin{array}{l} f_X : T \to [0,1] \text{ given by} \\ f_X(t) = \mathbb{P}(X = t) \end{array} \right.$

  Distribution of X

- In definition 3.1.5 above we assumed that S was Countable (finite) but that is not necessary.

3.1.1 Common Distributions :-

Definition 3.1.6. let X, Y be two discrete random variables have equal distribution if they have the same probability mass function. ie

$$f_X : T \to [0,1] \ \& \ f_Y : T \to [0,1]$$

$$- \quad f_X(t) = f_Y(t)$$

i.e. $\mathbb{P}(X=t) = \mathbb{P}(Y=t)$ $\forall \, t \in T$

$\boxed{X : S \to T}$ $\boxed{(S, \mathcal{F}, \mathbb{P})}$

$X \sim$ Uniform$(\{1,2,\dots,n\})$: $n \geq 1$; X is random variable s.t

$T := \text{Range}(X) = \{1,2,\dots,n\}$; $\mathbb{P}(X=k) = \frac{1}{n}$ $\forall \, k \in T$

$X \sim$ Bernoulli(p) $0 \leq p \leq 1$; $T = \{0,1\} \equiv \text{Range}(X)$

$$\left.\begin{aligned} \mathbb{P}(X=0) &= 1-p \\ \mathbb{P}(X=1) &= p \end{aligned}\right\}$$

takes concept of "Bernoulli trial"

1 - Success

0 - failure

$X \sim$ Binomial(n, p) : $n \geq 1$, $0 \leq p \leq 1$

$T \equiv \text{Range}(X) = \{0, 1, 2, \ldots, n\}$

$$\mathbb{P}(X=k) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k} \qquad k = 0, 1, 2, \ldots, n \text{ or } k \in T$$

$X \sim \text{Geometric}(p)$ $\quad 0 \leq p \leq 1$

$T = \text{Range}(X) = \mathbb{N}$

$$\mathbb{P}(X=k) = (1-p)^{k-1}\, p \qquad k \in T$$

$X \sim \text{Negative Binomial}(r, p)$ $\quad r \geqslant 1, \ 0 \leq p \leq 1$

$T \equiv \text{Range}(X) = \{k \in \mathbb{N} \mid k \geqslant r\}$

$$\mathbb{P}(X=k) = \binom{k-1}{r-1} p^{r} (1-p)^{k-r} \quad ; k \in T$$

$X \sim \text{Hypergeometric}(N, r, m)$ $\quad N, r, m \in \mathbb{N} \quad r < N, \ m < N.$

$T \equiv \text{Range}(X) = \{\min\{m, r\}, \ldots, \max\{0, m-(N-r)\}\}$

$$\mathbb{P}(X=k) = \frac{\binom{r}{k}\binom{N-r}{m-k}}{\binom{N}{m}}$$

Question:- • Experiment :- Toss a coin with probability of heads – p ; $0<p<1$; till we obtain the first head. Note the down the toss at which head occurs.

• Repeat Experiment 100 times [Each "time" experiment independently formed)

Q:- What is the likelihood that it takes at least 4 tosses in each of 100 times to obtain the 1st head ?

Definition 3.2.1 : Two random variables X, & Y are independent for if $(X \in A)$ and $(Y \in B)$ are independent for every event A in range of X and every event B in range of Y.

ie $X: S \to T_1$ $Y: S \to T_2$ S- equipped with probability $\mathbb{P}$

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$$

$A \subseteq T_1$
$B \subseteq T_2$

Ex: Observation: X, Y are discrete random variables ie. T_1, T_2 are countable (finite)

If $\mathbb{P}(X=s, Y=t) = \mathbb{P}(X=s)\, \mathbb{P}(Y=t)$ $\forall\, s \in T_1,\ t \in T_2$

then $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$ $\forall\, A \subseteq T_1, B \subseteq T_2$

X & Y are independent

Definition 3.2.3: A collection of random variables $X_1, X_2, \ldots, X_n$ are mutually independent if $\{X_j \in A_j\}_{j=1}^{n}$ are mutually independent for all events A_j in the range of X_j.

An arbitrary collection of $\{X_i\}_{i\in I}$, I - index set are independent if every finite sub-collection is mutually independent.

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Return to Question

Question:- • Experiment :- Toss a coin with probability of heads - p; $0<p<1$; till we obtain the first head. Note the down the toss at which head occurs.

$\boxed{X \sim \text{Geometric}(p)}$

INDEPENDENT $\boxed{X_1, X_2, \ldots X_{100}}$ — $X_i \sim \text{Geometric}(p)$

• Repeat Experiment $\boxed{100 \text{ times}}$ [Each "time" experiment independently turned]

Q:- What is the likelihood that it takes at least 4 tosses in each of 100 times to obtain the 1st head ?

$\mathbb{P}(X_1 \geq 4, X_2 \geq 4, \ldots X_{100} \geq 4)$?

Q: $X_1, \dots, X_{100}$; $X_i \sim \text{Geometric}(p)$

$X_1, \dots, X_{100}$ are independent

$$\mathbb{P}(X_1 \geq 4, X_2 \geq 4, \dots, X_{100} \geq 4) = ?$$

A:

- $\mathbb{P}(X_i = k) = \mathbb{P}(X_1 = k) = p(1-p)^{k-1}, \quad k \in \mathbb{N}$

$$\boxed{X_i \sim \text{Geometric}(p) \quad i = 1, \dots, 100} \xleftrightarrow{(*)} \boxed{\mathbb{P}(X_i \in A) = \mathbb{P}(X_1 \in A) \quad i = 2, \dots, 100}$$

- $$\mathbb{P}(X_1 \geq 4, X_2 \geq 4, \dots, X_{100} \geq 4) \underset{\text{independent}}{=} \prod_{k=1}^{100} \mathbb{P}(X_i \geq 4) \overset{(*)}{=} \left[\mathbb{P}(X_1 \geq 4)\right]^{100}$$

- $$\mathbb{P}(X_1 \geq 4) = \mathbb{P}\left(\bigcup_{k=4}^{\infty} (X_1 = k)\right) \overset{\text{Axiom 3}}{=} \sum_{k=4}^{\infty} \mathbb{P}(X = k) = (1-p)^3$$

Analysis I

$$T_n = \sum_{k=4}^{n} \mathbb{P}(X = k) = \sum_{k=4}^{n} p(1-p)^{k-1}$$

Ex:

$T_n \longrightarrow \alpha$ as $n \to \infty$

$\alpha = (1-p)^3$

$\mathbb{P}$ (that it takes at least 4 tosses in each of 100 times to obtain the 1st head ?)

$$= \mathbb{P}(X_1 \geq 4, X_2 \geq 4, \dots, X_{100} \geq 4)$$

$X_i \sim \text{Geometric}(p)$ $i = 1, \dots 100$
independent

$$= \left[\mathbb{P}(X_1 \geq 4)\right]^{100} = \left[(1-p)^3\right]^{100}$$

$$= (1-p)^{300}$$

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