Recall :-

$S = \{1,2,\dots n\}$ - Uniform $\{1,2,\dots n\}$ - Equally likely outcomes

$S = \{S, F\}$ - Bernoulli (p) - independent trials < success / failure

$S = \{0,1,2,\dots n\}$ - Binomial (n,p) - # of successes in n Bernoulli (p) trials

$S = \mathbb{N}$ - Geometric (p) - trial at which 1st success happens

$S = \mathbb{N} \cup \{0\}$ - Poisson(λ) - $n >>$ $np \approx \lambda$; limit $n \to \infty$

$S = \{0, \dots, \min\{r,m\}\}$ - Hypergeometric (N, r, m) - "Sampling m people without replacement from a population N, r- of which had a characteristic of interest.

Comment: Can we maintain one single sample space & define functions on it whose outputs provide the questions that we are interested in?

- Functions are called Random variables

S-countable/finite

3 - $\boxed{\text{Discrete}}$ Random Variables :-

3.1 - Random variables as functions

Example 3.1 : Suppose a fair coin is flipped 3 times.

① How many flips will come up heads?

② which will be the first flip (if any) that shows heads?

- and Probabilities of events in ① & ②.

Consider the natural sample space for experiment

$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$

X - be the function that describs total # of heads in 3 tosses, ie $X: S \to \{0,1,2,3\}$

Y - be the function that describs the 1st flip that produces heads, ie $Y: S \to \{1,2,3\} \cup \{None\}$

$\omega \in S$	$X(\omega)$	$Y(\omega)$
hhh	3	1
hht	2	1
hth	2	1
htt	1	1
thh	2	2
tht	1	2
tth	1	3
ttt	0	None

Q1 :-

$\mathbb{P}$(two heads in 3 tosses)

$= \mathbb{P}(\{\omega \in S \mid X(\omega) = 2\})$

$= \mathbb{P}(X^{-1}(\{2\}))$

$= \mathbb{P}(\{hht, hth, thh\})$

$= \frac{3}{8}$

Q2 :-

$\mathbb{P}$(first head is on the 3rd flip)

$= \mathbb{P}(\{\omega \in S \mid Y(\omega) = 3\})$

$= \mathbb{P}(Y^{-1}(\{3\}))$

$= \mathbb{P}(\{tth\})$

$= 1/8$

$\mathbb{P}(X = 0) = \mathbb{P}(\{ttt\}) = 1/8$

$\mathbb{P}(Y = 1) = \mathbb{P}(\{hht, hth, htt, hhh\}) = \frac{4}{8} = \frac{1}{2}$

Similarly, $\mathbb{P}(X=1) = 3/8$ $\quad \mathbb{P}(X=3) = 1/8$

$\mathbb{P}(Y=2) = 2/8$ $\quad \mathbb{P}(Y = none) = 1/8$

- Provides a complete describe of how X & Y distribute probabilities onto there range. - In both cases only a single sample space was required. Two different questions were approached by defining functions on that single sample space.

S - Countable or finite

Theorem 3.1.2 :- let S be a sample space with probability $\mathbb{P}$ and let $X: S \to T$ be a function. Then X generates a probability $\mathbb{Q}$ on T given by:

$$\mathbb{Q}(B) = \mathbb{P}(X^{-1}(B)) \qquad \text{for } B \subseteq T.$$

The probability $\mathbb{Q}$ is called the "distribution of X" since it describes how X distributes the probability from S onto T.

Proof :- let $\mathcal{F} = \mathcal{P}(S)$ power set associated to S

$\mathcal{F}_2 = \mathcal{P}(T) \equiv$ power set associated to T

$\therefore \mathbb{Q} : \mathcal{F}_2 \to [0,1)$ (✓)

$\mathbb{Q}(B) = \mathbb{P}(X^{-1}(B))$.

$X^{-1}(B) \in \mathcal{F}$ & $\mathbb{P} : \mathcal{F} \to [0,1)$ $\Big\}$ by definition

Axiom ① $\mathbb{Q}(T) = 1$

$$\mathbb{Q}(T) = \mathbb{P}(X^{-1}(T)) = \mathbb{P}(S) = 1$$

Axiom ② $\{B_i\}_{i \geq 1}$ $\quad B_k \cap B_\ell = \phi \quad k \neq \ell \quad B_j \subseteq T$

$$\mathbb{Q}\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \mathbb{Q}(B_j)$$

LHS $\quad \varphi\left(\bigcup_{j=1}^{\infty} B_j\right) = \mathbb{P}\left(X^{-1}\left(\bigcup_{j=1}^{\infty} B_j\right)\right)$

Observe:- $\quad X^{-1}\left(\bigcup_{j=1}^{\infty} B_j\right) = \left\{ \omega \in S \mid X(\omega) \in \bigcup_{j=1}^{\infty} B_j \right\}$

$$\overset{\text{Ex.}}{\underset{\text{Fact}}{=}} \bigcup_{j=1}^{\infty} \left\{ \omega \in S \mid X(\omega) \in B_j \right\}$$

$$= \bigcup_{j=1}^{\infty} X^{-1}(B_j)$$

$$= \mathbb{P}\left(\bigcup_{j=1}^{\infty} X^{-1}(B_j)\right)$$

$\frac{\text{Ex}}{\text{Fact}}$: $\{B_j\}_{j=1}^{\infty}$ are disjoint then $\{X^{-1}(B_j)\}_{j=1}^{\infty}$ are also disjoint.

$\mathbb{P}$ satisfies Axiom 2 $\quad = \sum_{j=1}^{\infty} \mathbb{P}(X^{-1}(B_j))$

$\left[T_n = \sum_{j=1}^{n} \mathbb{P}(X^{-1}(B_j)) \right.$ & $T_n \to T$ as $n \to \infty$

$\left. T := \sum_{j=1}^{\infty} \mathbb{P}(X^{-1}(B_j)) \right]$

$S_n = \sum_{j=1}^{n} \varphi(B_j)$ & $S_n \to T$ as $n \to \infty$

$$= \sum_{j=1}^{\infty} \varphi(B_j)$$

$\therefore \varphi$ satisfies Axiom 2.

$\square$