Recall :- **Discrete Random variable**

- $X : S \to T$ $\quad$ S – Countable set $\quad$ $\mathcal{F} = \mathcal{P}(S)$

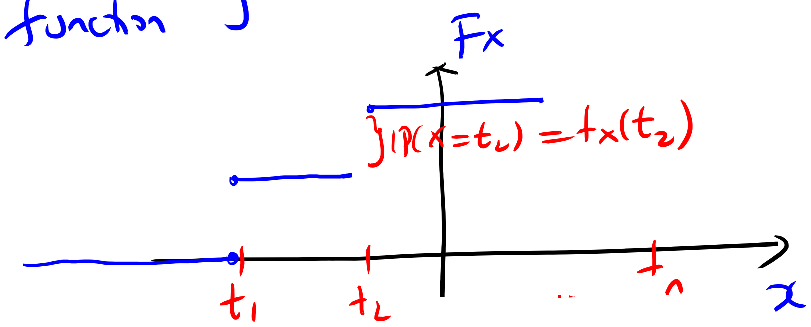
$\mathbb{P} : \mathcal{F} \to [0,1]$; $\mathbb{P}(S) = 1$ & $\{E_k\}_{k \geq 1}$ $\quad$ $E_k \cap E_j = \phi$

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

[p.m.f] : $f_X(t) = \mathbb{P}(X = t) \quad \forall t \in T$

[Distribution function] $\quad F_X : \mathbb{R} \to [0,1]$

$$F_X(x) = \mathbb{P}(X \leq x)$$
$$= \mathbb{P}(X^{-1}(-\infty, x])$$
$$= \sum_{\substack{t \in T \\ t \leq x}} \mathbb{P}(X = t)$$



- jump discontinuities & jumps are at $t \in T$

3.3.1. Functions of random variables [Discrete]

$X : S \to T$ $\quad$ discrete random variable

Question 1 :- $f : \mathbb{R} \to \mathbb{R}$ $\quad$ $Y = f(X)$. What is the distribution of Y?

Question 2 :- $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad$ $X_1 : S \to T_1$ $\quad$ $X_2 : S \to T_2$

$Z = f(X_1, X_2)$ $\quad$ What is the distribution of Z?

Generalize: $f : \mathbb{R}^n \to \mathbb{R}$ $\quad$ $X_i : S \to T_i$ $\quad$ $1 \leq i \leq n$

$Z = f(X_1, X_2, \ldots X_n)$ $\quad$ What is the distribution of Z?

Example 3.3.1 : • $X \sim \text{Uniform}(\{-2,-1,0,1,2\})$

• pmf. $f_X(t) = \mathbb{P}(X=t) = \frac{1}{|T|} = \frac{1}{5}$ for $t \in \{-2,-1,0,1,2\} = T$

• $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$.

$Y = g(X)$ Distribution of Y?

Range of Y : $U = g(X) = \{0, 1, 4\}$ $(\because g(x) = x^2)$

[Y is also a Discrete random variable $Y: S \to U$]

Pmf of Y : $\mathbb{P}(Y=0) = \mathbb{P}(g(X)=0) = \mathbb{P}(X=0) = \frac{1}{5}$

[$g(X)=0 \iff X^2=0 \iff X=0$]

$\mathbb{P}(Y=1) = \mathbb{P}(g(X)=1) = \mathbb{P}(X=1 \cup X=-1)$

[$g(X)=1 \iff X^2=1 \iff X=-1$ or $X=1$]

$= \mathbb{P}(X=1) + \mathbb{P}(X=-1) = \frac{1}{5} + \frac{1}{5} = 2/5$

$\mathbb{P}(Y=4) = \mathbb{P}(g(X)=4) = \mathbb{P}(X=2 \cup X=-2)$

[$g(X)=4 \iff X^2=4 \iff X=-2$ or $X=2$]

$= \mathbb{P}(X=2) + \mathbb{P}(X=-2) = \frac{1}{5} + \frac{1}{5} = 2/5$

$\square$

Example 3.3.3. Let $X, Y \sim$ Bernoulli (p) be two independent random variable.

$X: S \to \{0,1\}$; $Y: S \to \{0,1\}$; $\mathbb{P}(X=a, Y=b) = \mathbb{P}(X=a) \times \mathbb{P}(Y=b)$

$\mathbb{P}(X=1) = p$; $\mathbb{P}(Y=1) = p$; independence

$\mathbb{P}(X=0) = 1-p$; $\mathbb{P}(Y=0) = 1-p$; $b, a \in \{0,1\}$

Consider $h: \mathbb{R}^2 \to \mathbb{R}$, given by: $h(x,y) = x+y$

Q: $Z = h(X,Y) = X+Y$. What is the distribution of Z?

Intuitive Answer: $X \equiv$ # of heads in one toss of a biased coin (p)

(independent)

$Y \equiv$ # of heads in one toss of a biased coin (p)

$Z = X+Y :\equiv$ # of heads in two tosses of a biased coin (p)

$\therefore Z \sim$ Binomial $(2,p)$

Answer:- $Z = h(X,Y) = X+Y$

Range$(X) = \{0,1\} =$ Range(Y) $\Rightarrow$ Range$(Z) = \{0,1,2\}$

[Z - Discrete random variable]

$\mathbb{P}(Z=0) = \mathbb{P}(h(X,Y)=0) = \mathbb{P}(X+Y=0)$

[$X+Y=0 \iff X=0, Y=0$]

Pmf of Z

$= \mathbb{P}(X=0, Y=0) \underset{\text{independence}}{=} \mathbb{P}(X=0)\,\mathbb{P}(Y=0)$

$= (1-p)^2$ ... (*)

Similarly $\mathbb{P}(Z=2) = \mathbb{P}(X=1, Y=1) = \mathbb{P}(X=1)\,\mathbb{P}(Y=1)$

$= p^2$ ... (**)

$$\mathbb{P}(Z=1) = \mathbb{P}((X=1, Y=0) \cup (X=0, Y=1))$$

$X+Y=1 \iff \{X=1, Y=0\}$ or $\{X=0, Y=1\}$

$$= \mathbb{P}(X=1, Y=0) + \mathbb{P}(X=0, Y=1)$$

$$= \mathbb{P}(X=1)\mathbb{P}(Y=0) + \mathbb{P}(X=0)\mathbb{P}(Y=1)$$

$$= p(1-p) + (1-p)p$$

$$= 2p(1-p) \quad \cdots \quad (***)$$

From $(*)$, $(**)$, $(***)$

$\therefore$

$$\mathbb{P}(Z=k) = \begin{cases} p^2 & k=2 \\ 2p(1-p) & k=1 \\ (1-p)^2 & k=0 \end{cases}$$

$\therefore \quad Z \sim \text{Binomial}(2, p) \qquad \square$

Methodology used above:

$X: S \to T \qquad Y: S \to U$ & independent

$T, U \subseteq \mathbb{N} \cup \{0\}$

$Z = X+Y \quad \cdots\cdots \to \quad \text{Range}(Z) \subseteq \mathbb{N} \cup \{0\}$

$n \in \text{Range}(Z) \qquad \mathbb{P}(Z=n) = ?$

$Z=n \iff X+Y=n$

$\iff \bigcup_{j \in \{0,1,\dots,n\}} \{X=j, Y=n-j\}$

$$\mathbb{P}(Z=n) = \mathbb{P}\left(\bigcup_{j=0}^{n} \{X=j, Y=n-j\}\right)$$

mutually exclusive

$$= \sum_{j=0}^{n} \mathbb{P}(\{X=j, Y=n-j\})$$

independence

$$= \sum_{j=0}^{n} \mathbb{P}(X=j)\,\mathbb{P}(Y=n-j)$$

Convolution of $f_X(\cdot)$ & $f_Y(\cdot)$:-

$$\mathbb{P}(Z=n) = \sum_{j=0}^{n} \mathbb{P}(X=j)\,\mathbb{P}(Y=n-j)$$

i.e.

$$\boxed{f_Z(n) = \sum_{j=0}^{n} f_X(j)\, f_Y(n-j)}$$

Convolution :- $f_Z(n) = f_X * f_Y(n) \qquad n \in \mathbb{N} \cup \{0\}$

Example 3.3.4 (implement above methodology)

$X \sim \text{Poisson}(\lambda_1)$, independent, $Y \sim \text{Poisson}(\lambda_2)$

$\mathbb{P}(X=k) = \dfrac{e^{-\lambda_1}\lambda_1^k}{k!}$ $k \in \{0,1,2,\dots\}$

$\mathbb{P}(Y=k) = \dfrac{e^{-\lambda_2}\lambda_2^k}{k!}$ $k \in \{0,1,2,\dots\}$

ⓐ $Z = X+Y$: Find distribution of Z.

Independence

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\cdot\mathbb{P}(Y=y)$$

$$= \frac{e^{-\lambda_1}\lambda_1^x}{x!}\cdot\frac{e^{-\lambda_2}\lambda_2^y}{y!} \quad - (**)$$

$x, y \in \{0,1,2,\dots\}$

Above [Methodology]

$$\mathbb{P}(Z=n) = \sum_{j=0}^{n} \mathbb{P}(X=j)\,\mathbb{P}(Y=n-j)$$

$(**)$ -

$$= \sum_{j=0}^{n} \frac{e^{-\lambda_1}\lambda_1^j}{j!}\;\frac{e^{-\lambda_2}\lambda_2^{n-j}}{(n-j)!}$$

$$= e^{-\lambda_1}e^{-\lambda_2}\sum_{j=0}^{n} \frac{\lambda_1^j}{j!}\;\frac{\lambda_2^{n-j}}{(n-j)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!} \lambda_1^{j} \lambda_2^{n-j}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{j=0}^{n} \binom{n}{j} \lambda_1^{j} \lambda_2^{n-j}$$

Binomial Expansion

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^{n} \quad , n \in \{0,1,2,\dots\}$$

$$Z \sim \text{Poisson}(\lambda_1+\lambda_2)$$

Ex: $X_i \sim \text{Poisson}(\lambda_i)$ $i = 0,1,2,\dots k$ X_i - independent

$$Z = \sum_{i=1}^{k} X_i$$

$$\Rightarrow Z \sim \text{Poisson}\left(\sum_{i=1}^{k} \lambda_i\right)$$

Hint: Use induction

(b) Find conditional distribution of $X \mid Z$

$Z = n$; $\quad Z = X + Y$; $\Rightarrow X \in \{0, 1, \dots n\}$

$k \geq n+1$

, $n \in \mathbb{N}$ $\quad \mathbb{P}(X = k \mid Z = n) = 0$

$k \in \{0, 1, \dots n\}$

. $\quad \mathbb{P}(X = k \mid Z = n) = \dfrac{\mathbb{P}(X = k, Z = n)}{\mathbb{P}(Z = n)}$

$$= \frac{\mathbb{P}(X = k, X + Y = n)}{\mathbb{P}(Z = n)}$$

$$= \frac{\mathbb{P}(X = k, k + Y = n)}{\mathbb{P}(Z = n)}$$

$$= \frac{\mathbb{P}(X = k, Y = n - k)}{\mathbb{P}(Z = n)}$$

$$= \frac{\mathbb{P}(X = k)\, \mathbb{P}(Y = n - k)}{\mathbb{P}(Z = n)}$$

$$= \frac{\dfrac{\cancel{e^{-\lambda_1}}\,\lambda_1^{k}}{k!}\;\dfrac{\cancel{e^{-\lambda_2}}\,\lambda_2^{n-k}}{(n-k)!}}{\dfrac{\cancel{e^{-(\lambda_1+\lambda_2)}}\,(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!}\,\frac{\lambda_1^{k}\,\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \quad - \quad \#$$

$Z = X + Y$ $\quad$ $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$ $\Rightarrow$ $Z \sim \text{Poisson}(\lambda_1+\lambda_2)$

$$X \mid Z=n \sim \text{Binomial}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$\therefore\ \mathbb{P}(X=k \mid Z=n) = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}$$

$k \in \{0, 1, 2, \ldots n\}$

$\square$