Recall:

S – (uncountable) e.g. $S = [0,1]$

$\mathcal{F}$ – (σ field) $\equiv$ event $\quad S \in \mathcal{F}$; $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$; $E_k \in \mathcal{F}, k \geq 1 \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$

$S = \mathbb{R}$ $\quad \mathcal{F}$ – Smallest σ field containing all intervals

Borel sets $\quad A \in \mathcal{F}$ – interval, complement, unions / [combination]

[Probability] $\quad \mathbb{P} : \mathcal{F} \to [0,1]$ $\quad$ – Probability density function – $\int_{-\infty}^{\infty} f(x)\,dx = 1$

[12^th class] Def. $\quad f : \mathbb{R} \to [0,\infty)$; p.c. ;

$A \in \mathcal{F}$; $\quad \mathbb{P}(A) = \int_A f(x)\,dx$

- $\mathbb{P}(S) = \mathbb{P}(\mathbb{R}) = 1$ $\quad$ • – $\mathbb{P}(\{x\}) = 0 \quad \forall\, x \in \mathbb{R}$

- $X : S \to \mathbb{R}$ such that $\quad A$ – Borel sets

$(S, \mathcal{F}, \mathbb{P})$ $\quad X^{-1}(A) \in \mathcal{F}$ $\quad$ [X – random variable

- X is a continuous random variable

$$\mathbb{P}(X \in A) = \int_A f_X(x)\,dx$$

for some p.d.f $\quad f_X : \mathbb{R} \to [0,\infty)$

$\rightarrow$ connection with X.

- **Cummulative Distribution function** : $(S, \mathcal{F}, \mathbb{P})$

$X : S \to \mathbb{R}$ random variable

$F_X : \mathbb{R} \to [0,1]$ given by $\quad F_X(x) = \mathbb{P}(X \leq x) \equiv \mathbb{P}(X \in (-\infty, x])$

X is a Continuous random variable on $(S, \mathcal{F}, \mathbb{P})$

$X: S \to \mathbb{R}$ $\exists$ $f_X : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx.$$

[Distinguishing random variable from Discrete]

$$\mathbb{P}(X = a) = \int_a^a f_X(x)\, dx = 0$$

Theorem 5.2.5 :- Let X be a random variable with piecewise continuous density function $f_X : \mathbb{R} \to \mathbb{R}$. Then the distribution function $F_X : \mathbb{R} \to [0,1]$ [$F_X(a) = \mathbb{P}(X \leq a)$] is given

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\, dx$$

Moreover, where $f_X(\cdot)$ is continuous ; $F_X(\cdot)$ is differentiable and at all such point $a \in \mathbb{R}$ $F_X'(a) = f_X(a)$

Proof: [sketch]

- $F_X(a) = \int_{-\infty}^{a} f_X(x)\, dx$ $\rightarrow$ $f_X(\cdot)$ is p.c $\Downarrow$ $\boxed{F_X(\cdot) \text{ is continuous function}}$

- Fundamental Theorem of integral calculus

$$F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx \Longleftarrow F_X'(\cdot) = f_X(\cdot)$$

$\Longrightarrow$ at all continuity points of f

$\square$

Observations :-

- $F_X'(\cdot) = f_X(\cdot)$ whenever $f_X(\cdot)$ is continuous

- $\boxed{f_X \text{ is p.c}}$ $\Longrightarrow$ $F_X(\cdot)$ it also entirely determines the distribution of X.

[Proof is hard]

$$\mathbb{P}(X \in A) = \int_A f_X(x)\,dx \quad \forall A \in \mathcal{F} \ \text{(Borel set)}$$

5.2.1 Examples of Continuous Distributions

Definition 5.2.6 :- $X \sim \text{Uniform}(a,b)$ $\quad a<b \quad a,b \in \mathbb{R}$

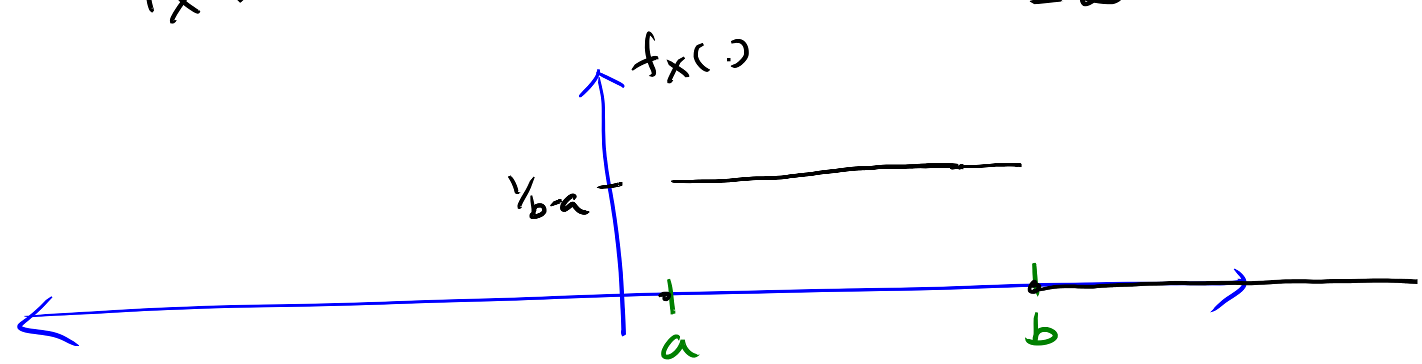
X is a continuous random variable with p.d.f given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a<x<b \\ 0 & \text{otherwise} \end{cases}$$

X is said to be uniformly distributed in (a,b).

Distribution function of $X \sim \text{uniform}(a,b)$

$$F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy$$



- $x \le a \Rightarrow f_X(y) = 0 \quad \forall \; y \le x$

$$\therefore \quad F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{-\infty}^{x} 0\,dy \overset{Ex}{=} 0$$

- $a < x < b \Rightarrow f_X(y) = \begin{cases} \frac{1}{b-a} & a < y \le x \\ 0 & y \le a \end{cases}$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{a}^{x} \frac{1}{b-a}\,dy$$

$$= \frac{y}{b-a}\Big|_{a}^{x}$$

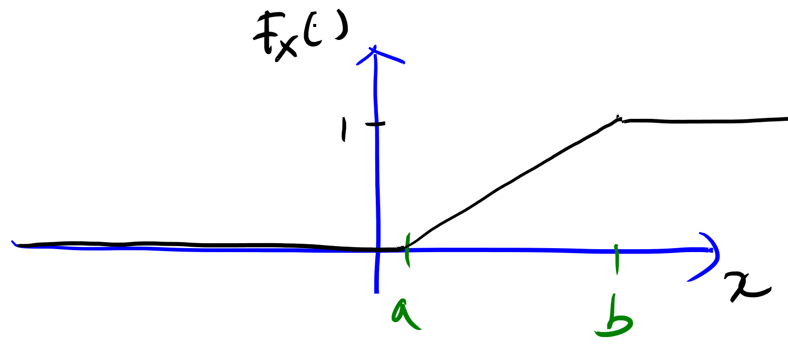
$$= \frac{x-a}{b-a}$$

- $x \geq b \Rightarrow f_X(y) = \begin{cases} 0 & y \le a \\ \frac{1}{b-a} & a < y < b \\ 0 & b \le y \le x \end{cases}$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{-\infty}^{a} 0\,dy + \int_{a}^{b} \frac{1}{b-a}\,dy + \int_{b}^{x} 0 \cdot dy$$

$$= \frac{y}{b-a}\Big|_{a}^{b} = \frac{b-a}{b-a} = 1$$

$F_X : \mathbb{R} \to [0,1]$ - $\quad F_X(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases}$



Definition 5.27 : $X \sim \text{Exponential}(\lambda)$ or $X \sim \text{Exp}(\lambda)$

Motivation :- - Radio active isotopes decay to a stable form.

- $N(0)$ atoms at time 0 ; φ: fraction of radioactive atoms left at time t ?

$$\frac{N(t)}{N(0)} \approx e^{-\lambda t} \quad \text{for some } \lambda > 0$$

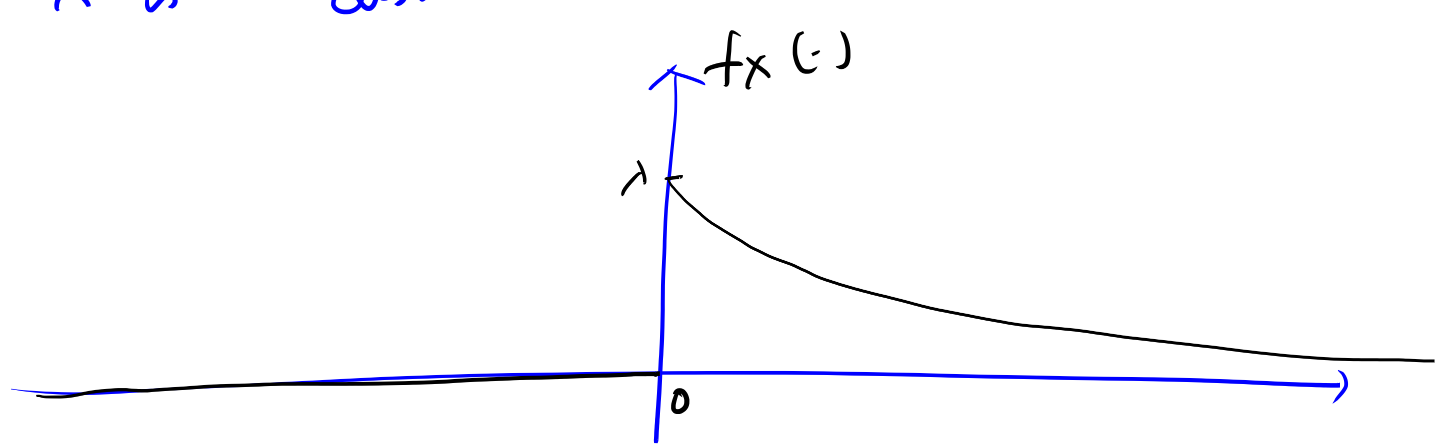
X - time taken by a "randomly chosen" radio active atom to decay to its stable form

Then $$\mathbb{P}(X \geq t) = e^{-\lambda t} \quad (\mathcal{F})$$

$X \sim \text{Exp}(\lambda)$ for $\lambda > 0$ if X is a continuous random variable with its probability density function given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

X is distributed as Exponential with parameter λ



$F_X : \mathbb{R} \to [0,1)$ $\quad F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy$

$\underline{x \le 0}$: $\quad f_X(y) = 0 \quad y \le x$

$$\Rightarrow \quad F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = 0 .$$

$\underline{x > 0}$ $\quad f_X(y) = \begin{cases} 0 & y \le 0 \\ \lambda e^{-\lambda y} & y \le x \end{cases}$

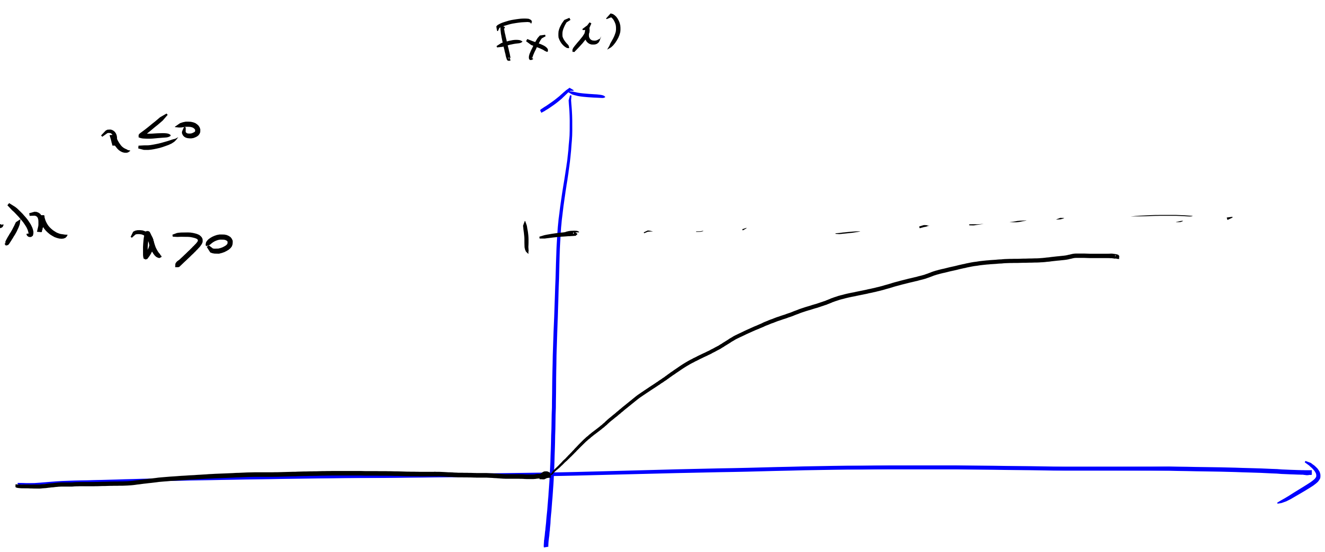
$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{-\infty}^{0} 0\,dy + \int_{0}^{x} \lambda e^{-\lambda y}\,dy$$

$$= \int_{0}^{x} \lambda e^{-\lambda y}\,dy$$

$\left[\begin{array}{l} g : [0,\infty) \to \mathbb{R} \\ g(y) = -e^{-\lambda y} \\ \text{Then } g'(y) = \lambda e^{-\lambda y} \end{array}\right]$

$$= -e^{-\lambda y}\Big|_{0}^{x} = 1 - e^{-\lambda x}$$

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$$



$t>0$ $\quad \mathbb{P}(X \geq t) = 1 - \mathbb{P}(X < t)$

[X is continuous random variable $\Rightarrow \mathbb{P}(X=t) = 0$]

$$= 1 - \mathbb{P}(X<t) - \mathbb{P}(X=t)$$

$$= 1 - [\mathbb{P}(X<t) + \mathbb{P}(X=t)]$$

$$= 1 - \mathbb{P}(X<t \cup X=t)$$

$$= 1 - \mathbb{P}(X \leq t)$$

$$= 1 - F_X(t)$$

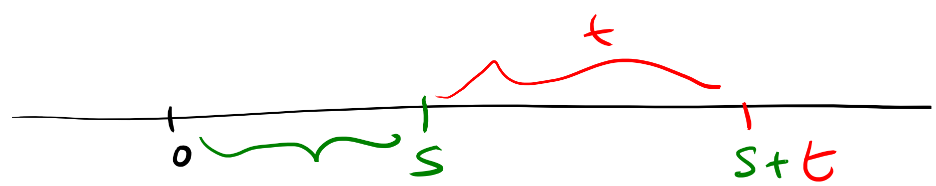
$$= 1 - (1 - e^{-\lambda t})$$

$$= e^{-\lambda t}$$

Verify ($\mp$) : for X

Memoryless Property :

$t>0 \qquad s>0$

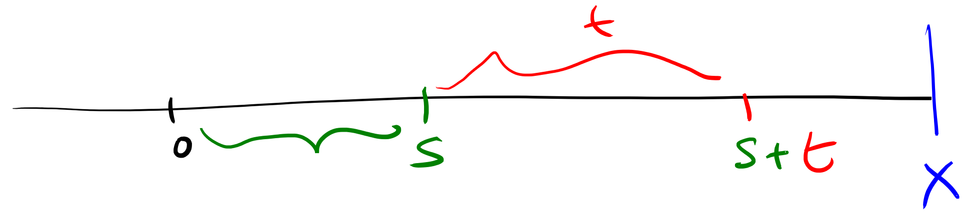


$X \sim Exp(\lambda)$ for $\lambda > 0$

$$\mathbb{P}(X > s + t \mid X > s)$$

[$\mathbb{P}(X>s) = \mathbb{P}(X \geq s) = e^{-\lambda s} > 0$]

[Conditional Probability Definition] $= \dfrac{\mathbb{P}(X > s+t \cap X > s)}{\mathbb{P}(X > s)}$



$$= \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > s)}$$

$$= \frac{\mathbb{P}(X \geq s+t)}{\mathbb{P}(X \geq s)} = \frac{e^{-\lambda(\cancel{s}+t)}}{e^{-\lambda \cancel{s}}}$$

$$= e^{-\lambda t} = \mathbb{P}(X \geq t)$$

$$= \mathbb{P}(X > t)$$

$$\mathbb{P}(X > s+t \mid X > s) = \mathbb{P}(X > t)$$

Discrete world

Geometric Random variable

Memoryless

Property

Continuous world

Exponential random variable

- other models :

- life time of a bulb $\equiv X$

- Waiting time in service $\equiv X$
  count

$X : S \to T$

S - countable
$\mathcal{F} = \mathcal{P}(S)$
$\mathbb{P} : \mathcal{F} \to [0,1]$
(Probability)

X is discrete random variable

$\boxed{\text{Distribution of } X}$: p.m.f. $f_X(t) = \mathbb{P}(X=t) \quad \forall t \in T$

$X : S \to \mathbb{R}$

S - countable
$\mathcal{F} = \mathcal{P}(S)$
$\mathbb{P} : \mathcal{F} \to [0,1]$
(Probability)

X is discrete random variable

$T = \text{Range}(X)$ - Countable

pmf $f_X(t) = \mathbb{P}(X=t) \quad \forall t \in T$

&

$\mathbb{P}(X=a) = 0 \quad \forall a \notin T$

Cummulative Distribution function of X : - [Discrete Random variable]

$F_X : \mathbb{R} \to [0,1)$

$F_X(x) = \mathbb{P}(X \le x)$

$$= \mathbb{P}(X^{-1}(-\infty, x])$$

Example: $T = \{0,1\}$ $\quad X \sim \text{Bernoulli}(p)$

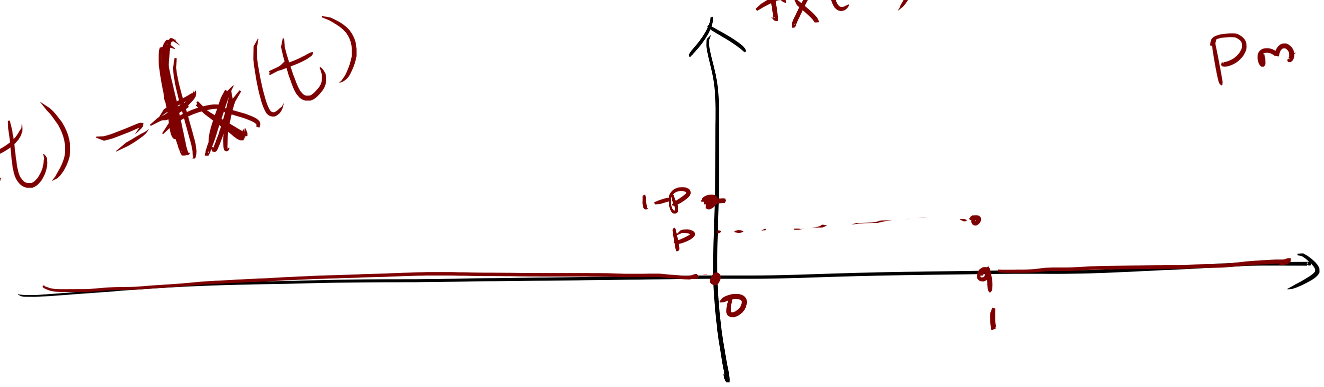
$$\mathbb{P}(X=1) = p \quad \& \quad \mathbb{P}(X=a) = 0 \ \ \forall a \notin T$$

$$\mathbb{P}(X=0) = 1-p$$

$$F_X(x) = \mathbb{P}(X \le x) \equiv \ ? \quad \forall x \in \mathbb{R}$$

$f_X(\cdot)$ pmf

$$\mathbb{P}(X=t) = f_X(t)$$



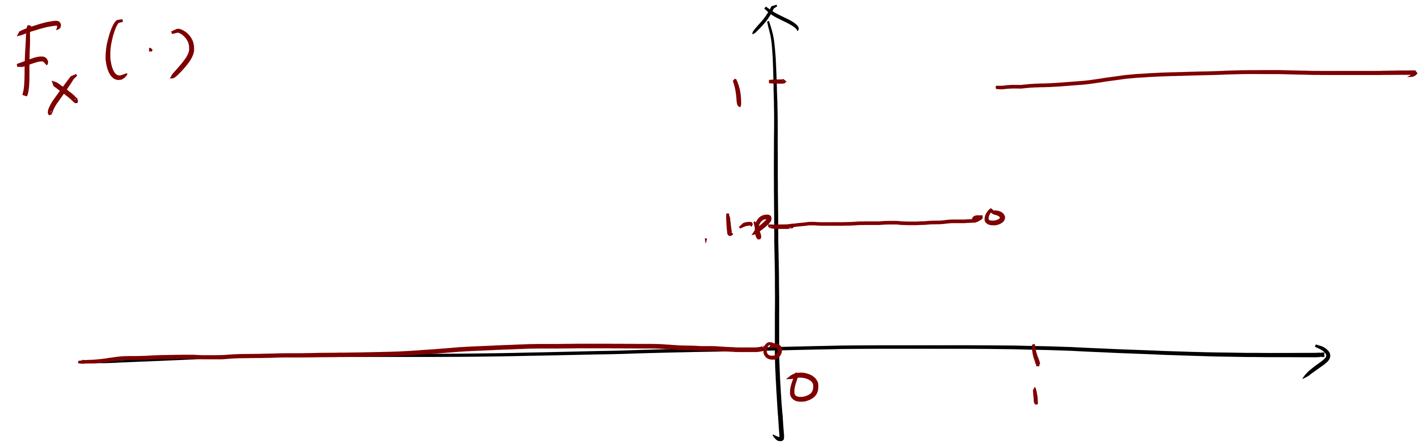
$x < 0$:- $\quad \mathbb{P}(X \le x) = 0$ as $X^{-1}(-\infty, x] = \phi$

$0 \le x < 1$ $\quad \mathbb{P}(X \le x) = \mathbb{P}(X^{-1}(-\infty, x]) = \mathbb{P}(\{0\}) = 1-p$

$x \ge 1$ $\quad \mathbb{P}(X \le x) = \mathbb{P}(X^{-1}(-\infty, x]) = \mathbb{P}(\{0,1\})$

$$= \mathbb{P}(\{0\}) + \mathbb{P}(\{1\})$$

$$= 1-p + p = 1$$



$F_X(\cdot)$ has jump discontinuities for discrete random variable X.

Definition [for our class]

$g : \mathbb{R} \to \mathbb{R}$ is piecewise continuous if $\exists\ n \geq 1$ & $\{t_k\}_{k=1}^{n}$ $t_k < t_{k+1}$

such that $g : (t_i, t_{i+1}) \to \mathbb{R}$ is a continuous function
$1 \leq i \leq n-1$