Recall:- Suppose we are interested in an event A occurring in an experiment. We want to estimate the value

$$p = \mathbb{P}(A) = \text{"chance of occurrence of } A\text{"}$$

- Repeat the experiment 'n' times, i.e. conduct n independent trials of experiment

[Estimate for p] $\hat{p} \equiv \dfrac{\#\text{ of times A occurs in } n \text{ trials}}{n} \equiv$ Relative frequency of A

- value of n ?
- How far is $\hat{p}$ from "true value" p ?

$X_1, X_2, \ldots, X_n$ - for each of n independent trials of the experiment

$$X_i = \begin{cases} 1 & \text{if A occurs} \\ 0 & \text{if A does not occur} \end{cases}$$

independent.

$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} = \overline{X}_n$$

$|\overline{X}_n - p| \equiv$ how large is this ?

observation: $X_i \sim \text{Bernoulli}(p)$;

Recall
$E X_i = p$
$Var(X_i) = p(1-p)$

$$\mathbb{P}(X_i = 1) = \mathbb{P}(A) = p$$

X_i is i.i.d Bernoulli(p)

8.2 Weak law of large numbers :-

Theorem:- Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d, ie independent and identically distributed random variables such that $EX_1 = \mu$ & $\mathrm{Var}(X_1) = \sigma^2 < \infty$

let $\varepsilon > 0$ be given.

$$\lim_{n \to \infty} \mathbb{P}\left(|\bar{X}_n - \mu| > \varepsilon \right) = 0$$

[Weak law of large numbers]

Proof:

$$\bar{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$E[\bar{X}_n] = E\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

linearity of Expectation

$$= \sum_{i=1}^{n} \frac{1}{n} \cdot E[X_i]$$

i.i.d.

$$= \frac{1}{n} \sum_{i=1}^{n} \mu$$

$$= \mu$$

$$\mathrm{Var}[\bar{X}_n] = \mathrm{Var}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

$\boxed{\mathrm{Var}[aZ] = a^2 \mathrm{Var}[Z]}$

$$= \frac{1}{n^2} \mathrm{Var}[X_1 + X_2 + \cdots + X_n]$$

$\boxed{\begin{array}{l} X, Y \text{ are independent} \\ \mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \end{array}}$ induction

$$= \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var}[X_i]$$

i.i.d

$$= \frac{\sum_{i=1}^{n} \sigma^2}{n^2}$$

$$= \frac{\sigma^2}{n}$$

Let $\varepsilon > 0$ be given.

$$\mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) = \mathbb{P}(|\bar{X}_n - \mu|^2 > \varepsilon^2)$$

Markov inequality to $Y = |\bar{X}_n - \mu|^2$, $\varepsilon^2 = c > 0$

$$\leq \frac{E|\bar{X}_n - \mu|^2}{\varepsilon^2}$$

$\mu = E\bar{X}_n$

$Var(\bar{X}_n) = E|\bar{X}_n - \mu|^2$

$$= \frac{Var[\bar{X}_n]}{\varepsilon^2}$$

$$= \frac{\sigma^2}{n\varepsilon^2}$$

$$\therefore \quad 0 \leq \mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} \cdot \frac{1}{n} \quad (*)$$

Analysis I: $\lim_{n\to\infty} \frac{1}{n} = 0$ & [Squeeze / Sandwich theorem]

$$(*) \Rightarrow \lim_{n\to\infty} \mathbb{P}(|\bar{X}_n - \mu| > \varepsilon) = 0$$

Going back to our motivational question:

$$p \equiv \frac{\#\text{ of times } A \text{ occurs in } n \text{ trials}}{n} = \text{Relative frequency of } A$$

$$X_i = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if otherwise} \end{cases} \qquad X_i \sim \text{Bernoulli}(p), \quad p = P(A)$$

Weak Law of Large Numbers

$\forall \varepsilon > 0$

$$\mathbb{P}(|\bar{X}_n - p| > \varepsilon) \longrightarrow 0 \quad \text{as } n \to \infty$$

ie $\forall \varepsilon > 0$

$$\mathbb{P}(|\hat{p} - p| > \varepsilon) \longrightarrow 0 \quad \text{as } n \to \infty$$

ie

$$\mathbb{P}(|\text{Relative frequency of } A - p| > \varepsilon) \longrightarrow 0 \quad \text{as } n \to \infty.$$

$\square$

Definition [Convergence in Probability]

Let $Y_1, Y_2, \ldots, Y_n, \ldots$ be a sequence of random variables. Let $\alpha \in \mathbb{R}$. We say "Y_n converges in Probability to α"

if $\forall \varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(|Y_n - p| > \varepsilon) = 0.$$

denoted by $Y_n \xrightarrow{p} \alpha$,

Weak law of large numbers :- $X_1, \ldots X_n, \ldots$ are i.i.d $\mathbb{E}[X_1] = p$, $Var[X_1] = \sigma^2 < \infty$

Then $\overline{X}_n \xrightarrow{p} p$ as $n \to \infty$.

ie $\forall \varepsilon > 0$ $\lim_{n\to\infty} \mathbb{P}(|\overline{X}_n - p| > \varepsilon) = 0$

Semester II. **Strong law of large numbers**

$X_1, \ldots X_n, \ldots$ are i.i.d $\mathbb{E}[X_1] = p$, $Var[X_1] = \sigma^2 < \infty$

Then $$\mathbb{P}\left(\lim_{n\to\infty} \overline{X}_n = p\right) = 1$$

~~**6.3 Moment generating functions** :-~~ [Sketch - Read section 6.3 & will be covered in Semester II]

X - discrete rv variable

Discussion holds for continuous Random variable as well

$X : S \to T$

- [Distribution of X] $\{\mathbb{P}(X=t)\}_{t \in T}$

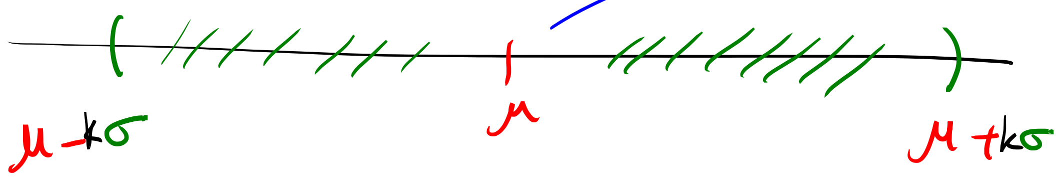
- $\mu = E[X] = \sum_{t \in T} t \, \mathbb{P}(X=t)$

- $\sigma^2 = \mathrm{Var}[X] = E[(X - E[X])^2]$

$$= \sum_{t \in T} (t-\mu)^2 \, \mathbb{P}(X=t)$$

"Essential Range of X"

Chebyshev inequality



Moments of X :

$k \geq 1$ $\quad m_k :=$ k^{th} moment of X $:= E[X^k]$

$$= \sum_{t \in T} t^k \, \mathbb{P}(X=t)$$

k^{th} moment is finite if $\uparrow$ converges absolutely

Result: "under certain conditions :- Carleman Condition"

$$\{m_k\}_{k \geq 1} \xleftrightarrow[\text{the distribution}]{\text{Determine}} \{\mathbb{P}(X=t)\}_{t \in T = \mathrm{Range}(X)}$$

Moment generating function :-

$$D = \{ t \in \mathbb{R} : E[e^{tX}] \text{ exists} \}$$

$M : D \to \mathbb{R}$ such that

$M(t) = E[e^{tX}]$ is called

moment generating function of X

Loose Calculation :- [Analysis I to Justify]

$x \in \mathbb{R}$, $\quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^k}{k!} + \cdots$

let $\delta > 0$ and $D = (-\delta, \delta)$, $m : D \to \mathbb{R}$

$$M(t) = E[e^{tX}]$$

$$= E\left[1 + tX + \frac{t^2X^2}{2} + \cdots + \frac{t^kX^k}{k!} + \cdots\right]$$

Analysis I ++ + linearity of Expectation

$$= 1 + tE[X] + \frac{t^2}{2}E[X^2] + \cdots + \frac{t^k E[X^k]}{k!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{t^k m_k}{k!}$$

(Convention $m_0 = 1$)

[Analysis I] :- $\quad M'(t) = \sum_{k=1}^{\infty} k t^{k-1} \frac{m_k}{k!}$

$$\Rightarrow \boxed{M'(0) = m_1}$$

[Analysis I] :- $\quad M''(t) = \sum_{k=2}^{\infty} k(k-1) t^{k-2} \frac{m_k}{k!}$

$$\Rightarrow \boxed{M''(0) = m_2}$$

[Analysis I] $\Rightarrow$ $m^{(k)}(0) = m_k \quad \forall k \geq 1$

o

Example 1:- $X \sim \text{Bernoulli}(p)$

$t \in \mathbb{R}$

$$E[e^{tX}] = e^{t\cdot 1} P(X=1) + e^{t\cdot 0} P(X=0)$$

$$= e^t p + 1(1-p)$$

$M : \mathbb{R} \to \mathbb{R}$ e $M(t) = 1-p+e^t p$

Example 2:- $X \sim \text{Binomial}(n,p)$

$t \in \mathbb{R}$,

$$E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$$

$$M(t) = (1-p+e^t p)^n$$

$X = X_1 + \dots + X_n$ where $X_i \sim \text{Bernoulli}(p)$ independent

$$E[e^{tX}] = E[e^{t\sum_{i=1}^{n} X_i}]$$

$$= E[\prod_{i=1}^{n} e^{tX_i}]$$

$X_1, \ldots, X_n$ independent then $e^{tX_1}, \ldots, e^{tX_n}$ are independent

$$= \prod_{i=1}^{n} E[e^{tX_i}]$$

$$= \prod_{i=1}^{n} (1-p+e^t p)$$

$$= (1-p+e^t p)^n$$

$\square$

OFFICE HOUR:

2:30 – 4 pm

Monday Feb $\underline{\underline{25}}$