Recall: X is a Discrete Random variable

$[T \subseteq \mathbb{R}]$ $X: S \to T$ S - Countable $f_X(t) = \mathbb{P}(X=t)$, $\forall t \in T$

- $E[X] = \sum_{t \in T} t\, \mathbb{P}(X=t) = \sum_{t \in T} t f_X(t)$

$|T| < \infty$ ✓ $|T| = \infty$ ↓ absolutely convergence required for $E[X] < \infty$

- $E[aX + bY] = a \underbrace{E[X]}_{<\infty} + b \underbrace{E[Y]}_{<\infty}$ $a, b \in \mathbb{R}$

- $X \geq 0 \Rightarrow E[X] \geq 0$

Example 4.1.13: $X \sim$ Bernoulli(p) $0 < p < 1$

$\mathbb{P}(X=1) = p$ & $\mathbb{P}(X=0) = 1-p$ $T = \{0, 1\}$

$$E[X] = 0 \cdot \mathbb{P}(X=0) + 1 \cdot \mathbb{P}(X=1)$$
$$= 0 \cdot (1-p) + 1 \cdot p = p$$

[X - need not assume the value E[X] with positive Probability]

Example 4.1.14: $X \sim$ Binomial(n, p)

$T = \{k : 0 \leq k \leq n\}$

$$E[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$
$$\overset{Ex}{=} np \quad \text{[induction]}$$

[$X_1 \sim$ Bernoulli(p), $X_2 \sim$ Bernoulli(p) independent

$\Rightarrow X = X_1 + X_2 \sim$ Binomial$(2, p)$

$E(X) = E[X_1] + E[X_2] = p + p = 2p$

$X \sim$ Binomial(n, p) Then $X = X_1 + \cdots + X_n$ $X_i \sim$ Bernoulli(p)

$E[X] = \sum_{i=1}^{n} E[X_i] = np$]

Example 4.1.15 $X \sim \text{Geometric}(p)$ $\quad 0<p<1$

$T = \mathbb{N}$ $\qquad \mathbb{P}(X=k) = (1-p)^{k-1} p \qquad k \in T$

$$E[X] = \sum_{k=1}^{\infty} k\, \mathbb{P}(X=k) \qquad [\, T_n \geq 0 \,]$$

$$\forall n \geq 1, \quad T_n = \sum_{k=1}^{n} k\, \mathbb{P}(X=k) = \sum_{k=1}^{n} k\, p(1-p)^{k-1} \overset{\text{Ex}}{=} \frac{1-(1-p)^n}{p} - n(1-p)^n$$

$(1-(1-p))$

[Analysis I :- $\lim_{n\to\infty} r^n = 0 \quad 0<r<1$; $\lim_{n\to\infty} n b^n = 0 \quad 0<b<1$]

$$\therefore \quad T_n \longrightarrow \frac{1}{p} \quad \text{as } n \to \infty.$$

$$\therefore \quad E[X] = \frac{1}{p} \qquad \square$$

4.14 $Y = f(X)$, $E[X] < \infty$ What is $E[Y] = ?$

Example 4.1.18 :-

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$Y = f[X]$

- $X \sim \text{Uniform}(\{-2,-1,0,1,2\})$ $\quad$ • $\mathbb{P}(X=k) = \frac{1}{5}$ $\quad k \in \{-2,-1,0,1,2\}$

Method I :-

- $Y = X^2$ $\quad$ • $\mathbb{P}(Y=0) = \frac{1}{5}$ & $\mathbb{P}(Y=4) = \frac{2}{5}$
  $\mathbb{P}(Y=1) = 2/5$

- $E[X] = \dfrac{-2 + -1 + 0 + 1 + 2}{5} = 0$ [not necessary]

- $E[Y] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 4 \cdot \frac{2}{5} = 2$

Distribution of Y
$\downarrow$
$E[Y]$

$Y = f(X) \qquad X \sim \text{uniform}\{-2, -1, 0, 1, 2\} \qquad f(x) = x^2$

Method 2

$$(f(X) = 4) = (X = 2 \cup X = -2)$$

- $$4 \cdot \mathbb{P}(f(X) = 4) = 4[\mathbb{P}(X=2) + \mathbb{P}(X=-2)]$$
$$= 2^2 \mathbb{P}(X=2) + (-2)^2 \mathbb{P}(X=-2)$$
$$= f(2)\mathbb{P}(X=2) + f(-2)\mathbb{P}(X=-2)$$

- $$0\,\mathbb{P}(f(X)=0) = 0\,\mathbb{P}(X=0) = f(0)\,\mathbb{P}(X=0)$$

- $$1\,\mathbb{P}(f(X)=0) = f(-1)\,\mathbb{P}(X=-1) + f(1)\,\mathbb{P}(X=1)$$

$$E[Y] = E[f(X)] = 4\,\mathbb{P}(f(X)=4) + 0 \cdot \mathbb{P}(f(X)=0) + 1 \cdot \mathbb{P}(f(X)=1)$$
$$= \sum_{k=-2}^{2} f(k)\,\mathbb{P}(X=k)$$

This leads to a result.

Theorem 4.1.19 Let $X : S \to T$ be a discrete random variable. $f : T \to U$. Then the expected value of $Y = f(X)$ may be computed as

$$E[f(X)] = \sum_{t \in T} f(t)\,\mathbb{P}(X=t) \quad - (*)$$

[We say Y has finite expectation if $(*)$ converges absolutely & ideally infinite & not defined as before]

Generalise :- X, Y are two random variables Discrete with joint p.m.f $p(x,y)$

$x \in \text{Range}(X)$
$y \in \text{Range}(Y)$

$g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$E[g(X,Y)] = \sum_{\substack{x \in \text{Range}(X) \\ y \in \text{Range}(Y)}} g(x,y)\, p(x,y)$$

[Recall: $Z = X + Y$
$E[Z] = E[X] + E[Y]$]

Chapter 6 Expectation of Continuous random variables

6.1 : $X : S \to \mathbb{R}$ is a continuous random variable

$A \in \mathcal{F}$ $\quad \mathbb{P}(X \in A) = \int_A f_X(x)\, dx$

where $f_X(\cdot)$ is p.d.f of X

[$f_X : \mathbb{R} \to [0,\infty)$, p.c & $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$]

Definition 6.1.1 : Let X be a continuous random variable with piecewise continuous p.d.f $f_X(\cdot)$, then Expected value of X is given by:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- provided the integral converges absolutely, (i.e. $\lim_{M,N \to \infty} \int_{-N}^{M} |x| f_X(x)\, dx < \infty$) is such a case we say that X has finite expectation

and if integral diverges to $\pm\infty$ then we say X has infinite expectation. If the integral diverges to a number which is not $\pm\infty$ then expectation does not exist.

Example 6.1.2. $X \sim \text{Uniform}(a,b)$, $-\infty < a < b < \infty$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

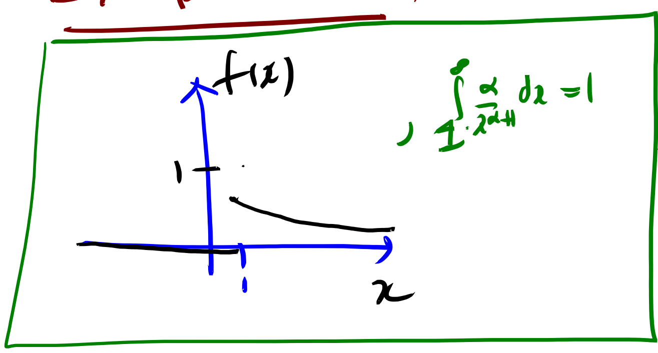
$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \overset{\text{Ex}}{=} \int_a^b x f_X(x)\,dx$$

$$= \int_a^b \frac{x}{b-a}\,dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{a+b}{2}$$

Example 6.1.3. $0 < \alpha < 1$ $\quad$ $X \sim \text{Pareto}(\alpha)$



$$f_X(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_1^\infty x \cdot \frac{\alpha}{x^{\alpha+1}}\,dx$$

$$= \lim_{M \to \infty} \int_1^M \alpha x^{-\alpha}\,dx$$

$M > 1$ $\qquad \int_1^M \alpha x^{-\alpha} dx = \alpha \left. \frac{x^{-\alpha+1}}{-\alpha+1} \right|_1^M = \frac{\alpha}{1-\alpha} M^{1-\alpha} - \frac{\alpha}{1-\alpha}$

$0 < \alpha < 1 \Rightarrow \underline{\underline{1-\alpha > 0}}$ $\quad \therefore \left[M \to \infty \Rightarrow M^{1-\alpha} \to \infty \right]$ Analysis

$$\therefore \int_{-\infty}^{\infty} x f_X(x)\, dx = \infty$$

$\therefore$ X has infinite expectation. $\square$

Theorem 6.1.5 X be a continuous random variable with p.d.f $f_X : \mathbb{R} \to [0, \infty)$

let $g : \mathbb{R} \to \mathbb{R}$ piecewise continuous; $Y = g(X)$ Then expected value of Y is given by

$$E[Y] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx.$$

as before $E[Y] < \infty$ if the integral converges absolutely.

Proof: [OMIT from syllabus] $\square$

Example 6.1.6 : (a) $T \sim \text{Exponential}(1/5)$ $\qquad f_T(t) = \begin{cases} \frac{1}{5} e^{-\frac{1}{5}t} & t \geq \\ 0 & \text{otherwise} \end{cases}$

$$E[T] = \int_{-\infty}^{\infty} t\, f_T(t)\, dt$$

$$= \int_0^{\infty} t \, \frac{1}{5} e^{-\frac{1}{5}t} \, dt = \lim_{N \to \infty} \int_0^N \frac{t}{5} e^{-\frac{1}{5}t} dt$$

$N \geq$

$$\int_0^N \frac{t}{5} e^{-\frac{1}{5}t} dt = \int_0^N t \, d(-e^{-\frac{1}{5}t}) = t(-e^{-\frac{1}{5}t})\Big|_0^N + \int_0^N 1 \, e^{-\frac{1}{5}t} dt$$

$$= -N e^{-\frac{1}{5}N} + \left(-\tfrac{1}{5}\right)^{-1} e^{-\frac{1}{5}t}\Big|_0^N$$

$$= -N e^{-\frac{1}{5}N} + 5 - 5 e^{-\frac{1}{5}N}$$

Analysis $N \to \infty$ $\longrightarrow$ $0 \quad + \; - \quad - 0$

$\therefore$ $T \sim \text{Exp}(1/5)$; $E[T] = 5$.

(b) $g : \mathbb{R} \to \mathbb{R}$ $\qquad g(t) = \max\{0, 1000 - 200t\}$

$Y = g(T)$

$$E[Y] = \int_{-\infty}^{\infty} g(t) \, f_T(t) \, dt$$

$$= \int_0^{\infty} \max\{0, 1000 - 200t\} \, \frac{1}{5} e^{-\frac{1}{5}t} \, dt$$

$\boxed{1000 - 200t > 0 \iff t < 5}$ $\qquad = \int_0^5 (1000 - 200t) \, \frac{1}{5} e^{-\frac{1}{5}t} dt$

$$= 200 \int_0^5 e^{-\frac{1}{5}t}\,dt - 200\int_0^5 t\,\frac{e^{-\frac{1}{5}t}}{5}\,dt$$

$$= 200\left((-5)\,e^{-\frac{1}{5}t}\Big|_0^5\right) - 200\left[\,t\left(-e^{-\frac{1}{5}t}\right)\Big|_0^5 + \int_0^5 e^{-\frac{1}{5}t}\,dt\right]$$

$$= 1000 - 1000\,e^{-1} - 200\left[-5e^{-1} + (-5)\,e^{-\frac{1}{5}t}\Big|_0^5\right]$$

$$= \cancel{1000} - \cancel{1000\,e^{-1}} + \cancel{1000\,e^{-1}} + 1000\left[e^{-1} \cancel{-1}\right]$$

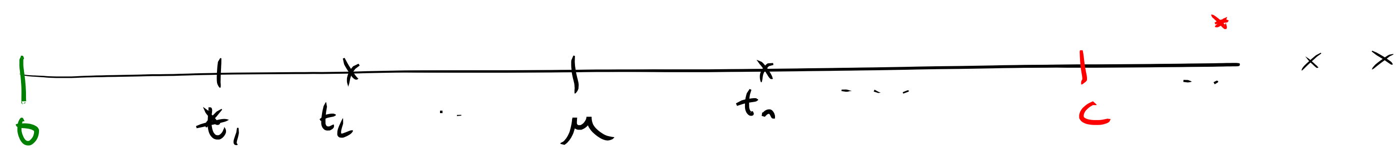
$$= 1000\,e^{-1}.$$

- Model for an insurance. $g \equiv$ payment

Markov's Inequality:

$X : S \to T \qquad f_X(t) = \mathbb{P}(X=t) \quad t \in T$

X is a non-negative discrete random variable, $E[X] = \mu$.



$\mathbb{P}(X \geq c) \equiv ?$ give bound on the probability

$$\mu = \sum_{t \in T} t\,\mathbb{P}(X = t)$$

$$\overset{Ex}{=} \underbrace{\sum_{\substack{t \in T \\ t < c}} t\, \mathbb{P}(X=t)}_{\substack{t \in T : t < c \\ t\, \mathbb{P}(X=t) \geq 0}} + \underbrace{\sum_{\substack{t \in T \\ t \geq c}} t\, \mathbb{P}(X=t)}_{t \in T : t \geq c}$$

$$\therefore \quad \mu \geq 0 + \underbrace{\sum_{\substack{t \in T \\ t \geq c}} t\, \mathbb{P}(X=t)}$$

$$\geq c \sum_{\substack{t \in T \\ t \geq c}} \mathbb{P}(X=t)$$

ie $\mu \geq c\, \mathbb{P}(X \geq c)$

$$\boxed{\begin{array}{l} X \geq 0, \text{ discrete r.v. then } c > 0 \\ \therefore \quad \mathbb{P}(X \geq c) \leq \dfrac{\mu}{c} \end{array}}$$

<u>Work with integrals</u>: X is continuous random variable with p.d.f

$f_X : \mathbb{R} \to [0, \infty)$ st $f_X(x) > 0 \quad \forall\, x \geq 0$

then

If $\mu = E[X]$, $c > 0$ $\Longrightarrow$ $\mathbb{P}(X \geq c) \leq \dfrac{\mu}{c}$