4- Summarising Discrete random variables :-

$X : S \to T$ S- Countable $\underbrace{f_X(t)}_{\text{Probability mass function}} = P(X=t) \quad \forall t \in T$

[Discrete random variable]

Recall :- Example 2.12 " On an average how many Bernoulli success will be there be after 'n' independent trials ? " [Develop a concept of average]

- Roll a die : What will be the average roll of a die

outcomes :- $\underbrace{1, 2, 3, 4, 5, 6}_{\text{Average of these}} = \frac{1+2+3+4+5+6}{6} = 3.5$

Rewrite :

$1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6)$

- results of all outcomes are added together after being weighted - weights being the probability of each outcome.

$\frac{X_1 + X_2 + \dots + X_n}{n}$:- Average $\xleftarrow{(w_i = 1/n)}$

:- Weighted Average $= \frac{w_1 X_1 + \dots + w_n X_n}{\sum_{i=1}^{n} w_i}$ $\rightsquigarrow$ $\boxed{\sum_{i=1}^{n} w_i = 1}$

Definition 4.1.1. Let $X : S \to T$ be a discrete random variable (T - Countable). Then the expected value of X is written as $E[X]$ and is given by

$$E[X] = \sum_{t \in T} t \, \mathbb{P}(X = t) \qquad - (*)$$

provided the sum in $(*)$ converges absolutely. In such a case we say X has "finite expectation". If the sum diverges to $\pm\infty$ we say the random variable X has "infinite expectation". If the sum diverges but not to infinity $(\pm)$, we say the random variable X does not expectation or expectation of X is not defined.

Back to our motivating example

Example:

- $X \sim \text{uniform}\{1, 2, \dots, n\}$
- $f_X(k) = \frac{1}{n} \qquad k \in \{1, 2, \dots, n\}$

$$E[X] = \sum_{k=1}^{n} k \, \mathbb{P}(X = k) = \frac{1}{n} \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \cdot \frac{1}{n} = \frac{n+1}{2}. \qquad \square$$

Example 4.1.4: $X : S \to T$

$$T = \{2, 4, 8, 16, \dots\} = \{2^k : k \in \mathbb{N}\}$$

$$\mathbb{P}(X = 2^k) = \frac{1}{2^k} \qquad k \in \mathbb{N}$$

clearly $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ - well defined Probability on T.

$$E[X] = \sum_{k=1}^{\infty} 2^k \, \mathbb{P}(X = 2^k)$$

let $T_n = \sum_{k=1}^{n} 2^k \, \mathbb{P}(X = 2^k) = \sum_{k=1}^{n} 2^k \cdot \frac{1}{2^k} = n$

[Analysis 1]

$\therefore \; T_n \to \infty$ as $n \to \infty$

$\therefore \; E[X]$ is infinite.

Example 4.1.5 : Suppose X is a random variable $X : S \to T$ $\quad T = \{(-1)^k 2^k \mid k \in \mathbb{N}\}$

$$f_X((-2)^k) = \mathbb{P}(X = (-2)^k) = \frac{1}{2^k} \qquad k \in \mathbb{N}$$

$$E[X] = \sum_{k=1}^{\infty} (-2)^k \, \mathbb{P}(X = (-2)^k)$$

$$T_n = \sum_{k=1}^{n} (-2)^k \, \mathbb{P}(X = (-2)^k) = \sum_{k=1}^{n} (-2)^k \cdot \frac{1}{2^k} = \sum_{k=1}^{n} (-1)^k$$

$\tilde{T}_n = \sum_{k=1}^{n} |(-1)^k| = n \to \infty$ as $n \to \infty$

(Series does not converge absolutely)

$$T_n = \begin{cases} -1 & n \in \text{odd} \\ 0 & n \in \text{even} \end{cases}$$

T_n diverges

So $E[X]$ is not defined, expectation of X does not exist. $\square$

Theorem 4.1.3 let c be real number
$X: S \to \mathbb{R}$ $\quad \mathbb{P}(X=c) = 1$

$$E[X] = c \quad \longleftrightarrow \quad E[c] = c$$

Convention

Proof:

$$E[X] = c \cdot \mathbb{P}(X=c) = c. \quad \square$$

Lemma 4.1.6:- let $X: S \to T$ be a discrete random variable

Then

$$E[X] < \infty \iff E[|X|] < \infty$$

(i.e $E[X]$ is a real number)

Proof:- let T be range of X.

$U = \{ |t| : t \in T \}$ - range of $|X|$

$$E[X] = \sum_{t \in T} t \, \mathbb{P}(X=t) \quad ; \quad E[|X|] = \sum_{u \in U} u \, \mathbb{P}(|X| = u)$$

To connect the two sums: $\quad \cdot \; \hat{T} = \{ t \in \mathbb{R} \mid |t| \in U \}$

- $\hat{T}$ - contains all element of T
- $t \in \hat{T}$ and $t \notin T \implies \mathbb{P}(X=t) = 0$

$$\therefore \; E[X] = \sum_{t \in T} t \, \mathbb{P}(X=t) \overset{(\text{Ex.})}{=} \sum_{t \in \hat{T}} t \, \mathbb{P}(X=t) \quad (*)$$

• $u \in \mathcal{U}$ $\quad \{|X|=u\} = \{X=u\} \cup \{X=-u\}$

$u \in \hat{T}$, $-u \in \hat{T}$

$$u\,\mathbb{P}(|X|=u) = u\,\mathbb{P}(X=u \cup X=-u)$$
$$\overset{\circ}{=} u\,\mathbb{P}(X=u) + u\,\mathbb{P}(X=-u)$$
$$= |u|\,\mathbb{P}(X=u) + |-u|\,\mathbb{P}(X=-u)$$

$(u=0,\ \mathbb{P}(|X|=0) \neq \mathbb{P}(X=0)+\mathbb{P}(-X=0))$

$$E[|X|] = \sum_{u\in\mathcal{U}} u\,\mathbb{P}(|X|=u) \overset{\circ,\,Ex}{=} \sum_{u\in\mathcal{U}} \underbrace{|u|\mathbb{P}(X=u) + |-u|\,\mathbb{P}(X=-u)}_{\text{non-negative}}$$

$$\overset{Ex}{=} \sum_{t\in\hat{T}} |t|\,\mathbb{P}(X=t)$$

$$= \sum_{t\in\hat{T}} |t\,\mathbb{P}(X=t)| \quad -(**)$$

As $E[X] = \sum_{t\in T} t\,\mathbb{P}(X=t)$, From ($*$) if $E[X]<\infty$ Then we know

R.H.S of ($**$) is finite $\Longleftrightarrow$ $E[|X|]<\infty$ $\square$

Theorem 4.17: Suppose X and Y are discrete random variables, both with finite expected value and both defined on some sample space S. Let $a,b\in\mathbb{R}$

① $E[aX] = a\,E[X]$

② $E[X+Y] = E[X] + E[Y]$

③ $E[aX+bY] = a\,E[X] + b\,E[Y]$

④ If $X \geq 0$ then $E[X] \geq 0$

Proof:- ② $X: S \to U \qquad Y: S \to V$

$T = \{u+v \mid u \in U,\ v \in V\}$

$\text{Range}(X+Y) = T \qquad Z = X+Y$

$Z: S \to T$

$$E[X+Y] = E[Z] = \sum_{t \in T} t \cdot \mathbb{P}(Z = t)$$

$$= \sum_{t \in T} t\, \mathbb{P}(X+Y = t)$$

Analysis I
absolutely converging series ;
Rearrange / Regroup is okay

$$\overset{(Ex.)}{=} \sum_{u \in U} \sum_{v \in V} (u+v)\, \mathbb{P}(X=u, Y=v)$$

$$\overset{(Ex.)}{=} \sum_{u \in U} u \sum_{v \in V} \mathbb{P}(X=u, Y=v) + \sum_{v \in V} v \sum_{u \in U} \mathbb{P}(X=u, Y=v)$$

Probability Ex $\displaystyle\sum_{u \in U} u \cdot \mathbb{P}(X=u) + \sum_{v \in V} v\, \mathbb{P}(Y=v)$

$$= E[X] + E[Y]$$

(4) $X \geq 0 \Rightarrow X: S \to T \qquad T \subseteq (0, \infty)$

$$E[X] = \sum_{t \in T} \underbrace{t\, \mathbb{P}(X=t)}_{\geq 0} \geq 0$$

Ex: (1) & (3) $\square$

Theorem 4.1.10 :- Suppose X and Y are discrete random variables both with finite expected value and defined on the same sample space. If X and Y are independent

then $$E[XY] = E[X]\, E[Y]$$

Proof: $X: S \to U \qquad Y: S \to V \qquad T = \{uv \mid u \in U, v \in V\}$

$Z = XY \qquad \text{Range}(Z) = T$

$$E[Z] = E[XY] = \sum_{t \in T} t\, \mathbb{P}(Z=t)$$

$$= \sum_{t \in T} t\, \mathbb{P}(XY=t)$$

Rearrangement is okay as series converges absolutely

$$\overset{Ex}{=} \sum_{u \in U} \sum_{v \in V} u v \, \mathbb{P}(X=u, Y=v)$$

Independence

$$= \sum_{u \in U} \sum_{v \in V} u v \, \mathbb{P}(X=u) \, \mathbb{P}(Y=v)$$

$$\overset{Ex}{=} \left(\sum_{u \in U} u \, \mathbb{P}(X=u) \right) \left(\sum_{v \in V} v \, \mathbb{P}(Y=v) \right)$$

$$= E[X] \, E[Y]$$

□

Example 4.1.11:

$X \sim \text{uniform}(\{1,2,3\})$ $\quad Y = 4 - X$

Ex: $Y \sim \text{uniform}(\{1,2,3\})$

Ex: $E[X] = \frac{3+1}{2} = 2$, $\quad E[Y] = 2$, $\quad E[X]\,E[Y] = 4$.

- $E[XY] = \frac{10}{3}$.

Because: $E[XY] = \sum_{t \in T} t \, \mathbb{P}(XY = t)$

$XY = X(4-X) = 4X - X^2 \in T = \{3, 4\}$

$$E[XY] = 3 \cdot \mathbb{P}(XY=3) + 4 \, \mathbb{P}(XY=4)$$

$$= 3 \, \mathbb{P}(X=1 \cup X=3) + 4 \, \mathbb{P}(X=2)$$

$$= 3\left(\tfrac{2}{3}\right) + 4\left(\tfrac{1}{3}\right) = \tfrac{10}{3} \neq 4$$

□