Recall

X is Discrete random variable $(S, \mathcal{F}, \mathbb{P})$

- $X : S \to T$ S - countable

$f_X(t) = \mathbb{P}(X = t)$ $t \in T$ $\}$ Distribution of X

Distribution function

it has jump discontinuities - atmost countable.

$\{ F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(X^{-1}(-\infty, x]) = \sum_{\substack{t \in T \\ t \le x}} f_X(t)$

- Transformations: $Y = g(X)$

$f_Y(t) = \mathbb{P}(g(X) = t) = \mathbb{P}(X \in g^{-1}(t))$

[E.g $X \sim$ Uniform $\{-2, -1, 0, 1, 2\}$ $Y = X^2$)]

$= \sum_{\substack{u \in T \\ g(u) = t}} \mathbb{P}(X = u)$

- X, Y are independent discrete random variables on $\mathbb{N} \cup \{0\}$

$Z = X + Y$

$f_Z(n) = \sum_{k=0}^{n} f_X(k) f_Y(n-k)$ $n \in \mathbb{N} \cup \{0\}$

— x — x —

Section 5.3 : Transformation of Continuous random variables.

Q : X is continuous random variable with p.d.f $f_X(\cdot)$

i.e $\mathbb{P}(a < X < b) = \int_a^b f_X(x)\, dx$

$Y = g(X)$ Find the distribution of Y.

$$\underbrace{F_X(x) = \mathbb{P}(X \le x)}_{\text{Distribution function}} = \int_{-\infty}^{x} f_X(y)\, dy$$

- certain properties for e.g. its continuous

& f is continuous $\Rightarrow F' = f$

Caution: X is discrete random variable then $X: S \to T$

$g: T \to \mathbb{R}$, $Y = g(X)$ is also a discrete random variable.

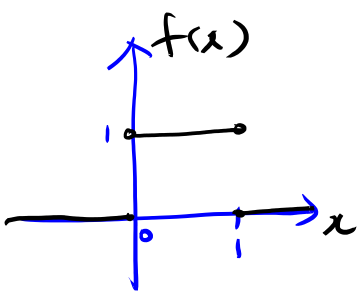
X is continuous random variable & $X: S \to \mathbb{R}$

$g: \mathbb{R} \to \mathbb{R}$, $Y = g(X)$ — does not immediately imply Y is continuous or discrete or neither.

$\boxed{\text{We need a standardised way to understand distribution of } Y.}$

Example 5.3.1

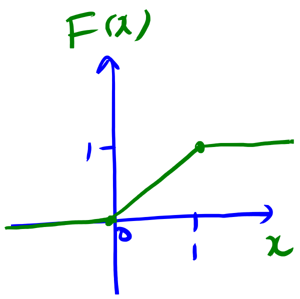
$X \sim \text{Uniform}(0,1)$



i.e. p.d.f

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

& $\mathbb{P}(a < X \leq b) = \int_a^b f_X(x)\,dx$



Recall. Distribution function

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Q: $Y = X^2$. Find "distribution" of Y

If Y discrete then find p.m.f.

neither: stop at distribution function of Y.

If Y is continuous then find p.d.f

Step 1 : Find distribution function of Y.

$$F_Y(y) = \mathbb{P}(Y \leq y) \qquad y \in \mathbb{R}.$$

$$= \mathbb{P}(X^2 \leq y) \quad \text{as } Y = X^2.$$

If $y < 0$ then $\{X^2 \leq y\} := X^{-1}((-\infty, y]) = \phi$

$$\therefore \mathbb{P}(X^2 \leq y) = \mathbb{P}(\phi) = 0$$

$$F_Y(y) = 0 \quad \text{if } y < 0 \quad - (*)$$

- If $y \geq 0$ then

$$F_Y(y) = \mathbb{P}(X^2 \leq y)$$

$$= \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx$$

$$= \begin{cases} \int_{-\sqrt{y}}^{0} 0\,dx + \int_{0}^{\sqrt{y}} 1\,dx & \text{— if } \sqrt{y} \leq 1 \iff \overset{0 \leq y}{y \leq 1} \\ \int_{-\sqrt{y}}^{0} 0\,dx + \int_{0}^{1} 1\,dx + \int_{1}^{\sqrt{y}} 0\,dx & \text{— if } \sqrt{y} > 1 \iff y > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} x \Big|_0^{\sqrt{y}} & 0 \le y \le 1 \\ x \Big|_0^{1} & y > 1 \end{cases}$$

$$\therefore \quad F_Y(y) = \begin{cases} \sqrt{y} & 0 \le y \le 1 \\ 1 & y > 1 \end{cases} \quad — (xx)$$

(*) and (xx)

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

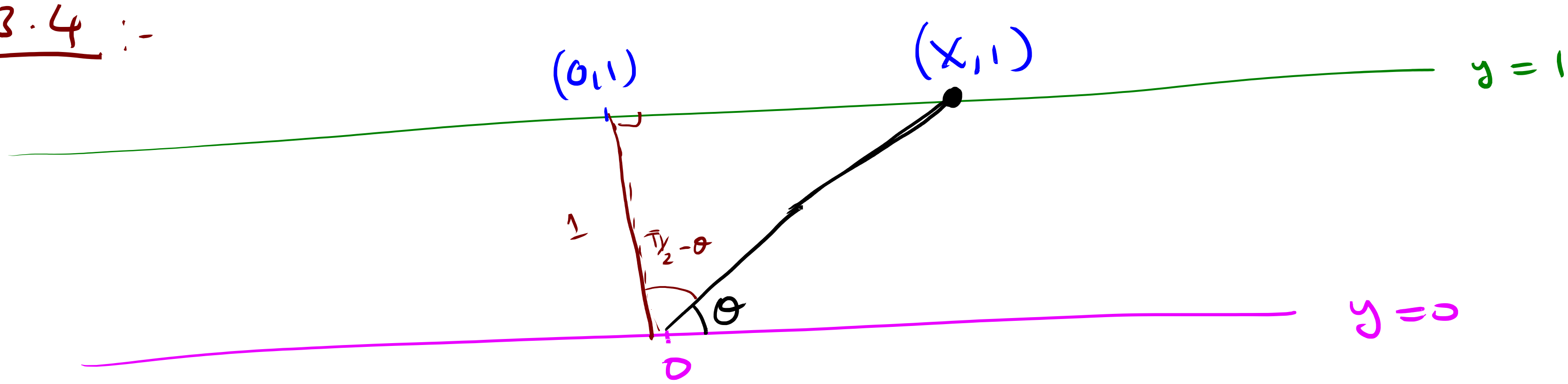
Step2:- Differentiating $F_Y(\cdot)$ [as its clearly a differentiable at all points except 0 and 1]

$$f_Y(y) = F_Y'(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & y > 1 \end{cases}$$

assume:- $f_Y(0) = 0$ $\quad f_Y(1) = 0$

Y has p.d.f given by $f_Y(y) = \begin{cases} 0 & \text{otherwise} \\ \frac{1}{2\sqrt{y}} & 0 < y < 1 \end{cases}$

Example 5.3.4 :-



- choose an angle in $(0, \pi)$ $\quad \theta \sim \text{Uniform}(0, \pi)$

What is the p.d.f. of X ?

$$X = \tan\left(\frac{\pi}{2} - \theta\right)$$

p.d.f. of θ :-
$$f_\theta(\theta) = \begin{cases} \frac{1}{\pi - 0} & 0 < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

Step 1: Find distribution function of X

$$F_X(x) = \mathbb{P}(X \le x)$$
$$= \mathbb{P}\left(\tan\left(\frac{\pi}{2} - \theta\right) \le x\right)$$
$$= \mathbb{P}\left(\frac{\pi}{2} - \theta \le \arctan(x)\right)$$
$$= \mathbb{P}\left(\theta \geq \frac{\pi}{2} - \arctan(x)\right)$$

Observe:
$$= 1 - \mathbb{P}\left(\theta < \frac{\pi}{2} - \arctan(x)\right)$$

$x \in \mathbb{R}$, $\frac{\pi}{2} - \arctan(x) \in (0, \pi)$

$$= 1 - \int_{-\infty}^{\pi/2 - \arctan(x)} f_\theta(\theta)\, d\theta$$

$$= 1 - \left[\int_{-\infty}^{0} 0\, d\theta + \int_{0}^{\pi/2 - \arctan(x)} \frac{1}{\pi}\, d\theta \right]$$

$$= 1 - \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan(x)\right)$$

$x \in \mathbb{R}$ $\quad F_X(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$

<u>step2</u>: Differentiate $F_X(\cdot)$ to get $f_X(\cdot)$

$$\therefore f_X(x) = F_X'(x)$$

$$= 0 + \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \qquad x \in \mathbb{R}.$$

Definition 5.3.5 $X \sim \text{Cauchy}(\theta, \alpha^2)$ $\theta \in \mathbb{R}$, $\alpha > 0$

If p.d.f of X is given by

$$f_X(x) = \frac{1}{\pi} \frac{\alpha}{\alpha^2 + (x-\theta)^2} \quad ; x \in \mathbb{R}$$

Then distribution function of F_X is given by

$$F_X(x) = \frac{1}{\pi} \arctan\left(\frac{x-\theta}{\alpha}\right)$$

Lemma 5.3.2: $a \neq 0$ and $b \in \mathbb{R}$. X is a continuous random variable with a continuous probability density function f_X.

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = ax + b$

$$Y = g(X)$$

Then Y is also continuous random variable variable with p.d.f.

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \quad y \in \mathbb{R}$$

Proof :-

Step 1: Find distribution function of Y

- Assume $a > 0$

$$F_Y(y) = \mathbb{P}(Y \leq y)$$
$$= \mathbb{P}(aX+b \leq y)$$
$$= \mathbb{P}(aX \leq y-b)$$
$$= \mathbb{P}\left(X \leq \frac{y-b}{a}\right) \quad \text{if } \underline{\underline{a>0}}$$
$$= F_X\left(\frac{y-b}{a}\right)$$

- Assume $a<0$

$$F_Y(y) = \mathbb{P}(Y \leq y)$$
$$= \mathbb{P}(aX+b \leq y)$$
$$= \mathbb{P}(aX \leq y-b)$$
$$= \mathbb{P}\left(X \geq \frac{y-b}{a}\right) \quad \text{if } \underline{\underline{a<0}}$$
$$= 1 - \mathbb{P}\left(X < \frac{y-b}{a}\right)$$

X-continuous

$$= 1 - \mathbb{P}\left(X \leq \frac{y-b}{a}\right)$$
$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

Step 2: Find p.d.f of Y by differentiating $F_Y(\cdot)$

$a>0$

$$F_Y(y) = F_X\left(\frac{y-b}{a}\right)$$

$$\therefore f_Y(y) = F_Y'(y)$$

chain rule

$$= F_X'\left(\frac{y-b}{a}\right) \frac{d}{dy}\left(\frac{y-b}{a}\right)$$

$$= f_X\left(\frac{y-b}{a}\right) \frac{1}{a}$$

$a<0$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = F_Y'(y)$$

chain rule

$$= 0 - F_X'\left(\frac{y-b}{a}\right) \cdot \frac{d}{dy}\left(\frac{y-b}{a}\right)$$

$$= -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$a \neq 0$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \quad \forall y \in \mathbb{R}$$

$\square$

5.2.2 Individual outcoms for continuous random variables

X – continuous random variable with p.d.f $f_X(\cdot)$

$\forall a \in \mathbb{R}$ $\quad \mathbb{P}(X=a) = \int_a^a f_X(x)\,dx = 0.$ $\square$

E.g. $X \sim$ uniform $(0,1)$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Take

$$g(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

. $f(x) = g(x)$ $x \in \mathbb{R}$ other than $f(0) \neq g(0)$ & $f(1) \neq g(1)$

. f, g are both density function & Probabilities i.e the same determined by f & g are

$A \in \mathcal{F}$; $\quad \mathbb{P}(A) = \int_A f(x)\,dx = \int_A g(x)\,dx$

Provided f and g disagree at almost countably many points

Convention :- • Think of $f \equiv g$ if they are density functions that differ at atmost finite / Countably many points.

– as they produce the same probabilities

• Previous calculation :- Differentiating $F_y(\cdot)$ we can forget finitely many points. to set $f_y(\cdot)$

[as we can alter the density function at finitely many / Countably many points and it won't affect the probabilities. !]

TOMORROW OFFICE HOUR

Thursday 4:30 pm to 6 pm.