Recall: S - uncountable space

$\mathcal{F}$ = σ-field $\equiv \begin{cases} \cdot\ S \in \mathcal{F}, \quad \cdot\ E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}; \\ \{E_k\}_{k \geq 1} \quad E_k \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{F} \end{cases}$

$\mathbb{P}: \mathcal{F} \to [0,1]$ ① $\mathbb{P}(S) = 1$;

② $\{E_k\}_{k \geq 1}$ $E_l \cap E_j = \phi$ $l \neq j$

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

$S = \mathbb{R}$, $\mathcal{F} \equiv$ smallest σ-field that contain all interval [Borel sets]

For this course: $\mathcal{F} \equiv \exists A$ A - either an interval or complement of an interval or countable unions of them or some combination

Definition 5.1.4:- let $f: \mathbb{R} \to \mathbb{R}$ is called a density function if f satisfies the following:

(i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

(ii) f is piecewise continuous ← for this course

(iii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Theorem 5.1.5 let $f : \mathbb{R} \to \mathbb{R}_+$ be a density function $S = \mathbb{R}$, $\mathcal{F}$ - Borel sets $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ be given by

$$\mathbb{P}(A) = \int_A f(x)\,dx \qquad \forall \; A \in \mathcal{F}$$

Then $\mathbb{P}$ defines a Probability on $\mathbb{R}$. Then f is called the "density function" for the Probability $\mathbb{P}$.

Proof:- $S = \mathbb{R}$

① $$\mathbb{P}(\mathbb{R}) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

f is a density function (iii)

Step 0:- $\mathbb{P} : \mathcal{F} \to [0,1]$

$$\mathbb{P}(A) = \int_A f(x)\,dx \quad \begin{cases} \geq 0 & f(\cdot) \geq 0 \\ \leq 1 = \int_{\mathbb{R}} f(x)\,dx \end{cases}$$

[$\mathbb{P}$- is well defined function]

Assume :-- $\forall\, A \in \mathcal{F}$ ...... $\int_A f(x)\,dx$ well defined objected

② $\{E_k\}_{k \geq 1}$ $E_\ell \cap E_j = \phi$

Sketch of Proof

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \int_{\bigcup_{k=1}^{\infty} E_k} f(x)\,dx$$

Assume Fact from integration

$$= \sum_{k=1}^{\infty} \int_{E_k} f(x)\,dx$$

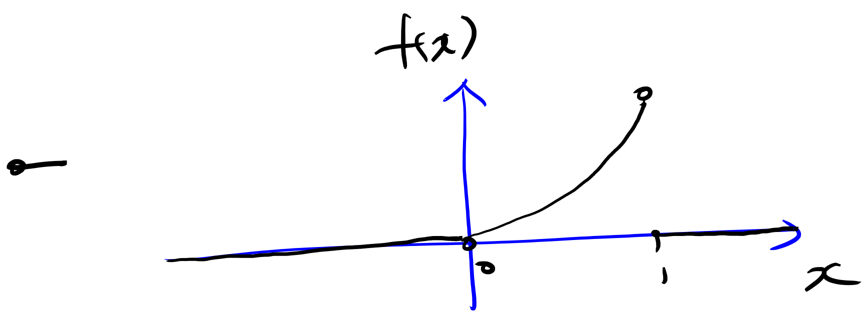
(Series equality)

$$= \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

$\mathbb{P}$ is indeed a Probability on $\mathbb{R}$. $\square$

Example 5.1.6 :- let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



- f is piecewise continuous with discontinuity point being 1
- $f(\cdot) \geq 0$ by definition
- $$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1^3 - 0^3 = 1$$

From Theorem 5.1.5

$\mathbb{P} : \mathcal{F} \to \mathbb{R}$ given by

$$\mathbb{P}(A) = \int_A f(x)\,dx$$ is a Probability on $\mathbb{R}$.

$A = [\frac{1}{5}, \frac{2}{5}] \subseteq [0,1]$

$$\mathbb{P}(A) = \int_{1/5}^{2/5} 3x^2\,dx = x^3 \Big|_{1/5}^{2/5} = \frac{7}{125}$$

$B = [\frac{3}{5}, \frac{4}{5}) \subseteq [0,1]$

$$\mathbb{P}(B) = \int_{3/5}^{4/5} 3x^2\,dx = x^3 \Big|_{3/5}^{4/5} = \frac{37}{125}$$

observation

1 [last class: [Density] Example: $f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$; $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{5}$ – uniform on $[0,1]$]

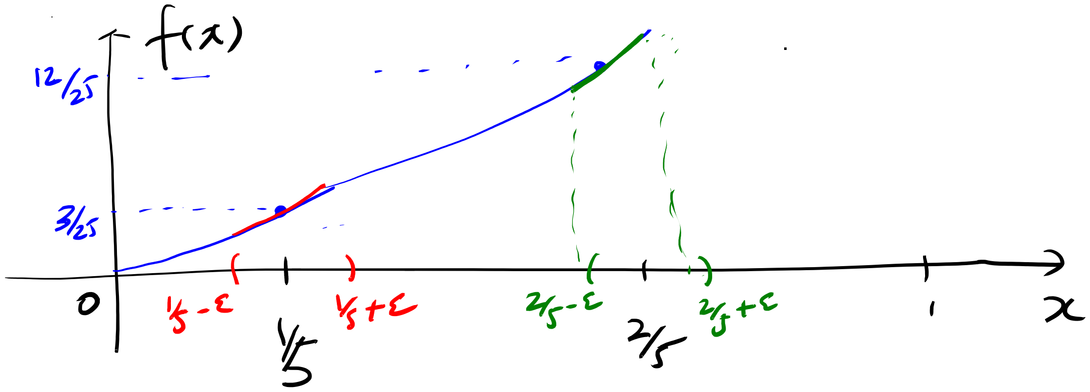
$\mathbb{P}$ given in this example is DIFFERENT from - uniform on $[0,1]$ example.

2 $$\mathbb{P}(\{a\}) = \int_a^a f(x)\,dx = 0$$

3 $$f\left(\frac{2}{5}\right) = 3\left(\frac{2}{5}\right)^2 = \frac{12}{25} = 4\left(f\left(\frac{1}{5}\right)\right) = 4\left(\frac{3}{25}\right)$$

But $\mathbb{P}(\{\frac{2}{5}\}) = \mathbb{P}(\{\frac{1}{5}\})$ (2) $= 0$

$\varepsilon > 0$



$$\mathbb{P}\left(\left[\tfrac{1}{5}-\varepsilon, \tfrac{1}{5}+\varepsilon\right]\right) = \int_{1/5-\varepsilon}^{1/5+\varepsilon} 3x^2\,dx \overset{Ex}{=} \frac{2}{25}\varepsilon + 2\varepsilon^3 \approx \frac{2}{25}\varepsilon$$

$$\mathbb{P}\left(\left[\tfrac{2}{5}-\varepsilon, \tfrac{2}{5}+\varepsilon\right]\right) = \int_{2/5-\varepsilon}^{2/5+\varepsilon} 3x^2\,dx \overset{Ex}{=} \frac{8}{25}\varepsilon + 2\varepsilon^3 \approx \frac{8}{25}\varepsilon$$

Because $f(\frac{2}{5}) = 4 f(1/5)$ $\dashrightarrow$ Any small interval around $2/5$ will have a probability 4 times probability of a similar size interval around $1/5$.

5.2 Continuous Random variables

X is a continuous random variable — has some restriction compared to a discrete random variable.

Definition 5.2.1: let $(S, \mathcal{F}, \mathbb{P})$ be a Probability space. Let $X: S \to \mathbb{R}$ be a function. Then X is a random variable if whenever B – Borel set in $\mathbb{R}$ then $X^{-1}(B) \in \mathcal{F}$. – (*)

- in the discrete setting this is met trivially
- for the sake of this course, we will only consider "nice" function that satisfy the criterion
  - in particular we will do continuous random variable

Definition 5.2.2:- let $(S, \mathcal{F}, \mathbb{P})$ be a Probability space. A random variable $X: S \to \mathbb{R}$ is called a Continuous random variable if there is a density function $f_X: \mathbb{R} \to \mathbb{R}$ such that for any event A in $\mathbb{R}$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\,dx.$$

$f_X(\cdot)$ is called the Probability density function of X.

Recall: S - Countable $X : S \to T$ discrete random variable.

Probability mass function of X $:- f_X(t) = \underbrace{\mathbb{P}(X=t)}_{X \text{ distributes Probabilities onto } T.}$ $\forall t \in T$

Lemma 5.2.3 Let $(S, \mathcal{F}, \mathbb{P})$ be a Probability space & X be a continuous random variable from $S \to \mathbb{R}$ with p.d.f given by $f_X : \mathbb{R} \to \mathbb{R}_+$ then

$$\forall a \in \mathbb{R} \quad \mathbb{P}(X=a) = 0.$$

Proof:

let $a \in \mathbb{R}$

$$\mathbb{P}(X=a) = \mathbb{P}(X \in [a,a]) = \int_a^a f(x)\,dx = 0$$

$\square$

Q: Why X is called discrete random variable?
X is called continuous random variable?

Answer to Q $\xleftrightarrow{\text{AND}}$ Differences between them

① $X : S \to \boxed{T}$ Discrete $\qquad \mathbb{P}(X \in A) = \sum_{t \in A} \mathbb{P}(X=t)$ $\qquad$ A-Countable

$\boxed{\text{Countable}}$

(2) $X : S \to \mathbb{R}$ A - event in $\mathbb{R}$; $\mathbb{P}(X \in A) = \int_A f_X(x)\,dx$.

Continuous random variable with p.d.f $f_X : \mathbb{R} \to \mathbb{R}_+$

Definition 5.2.4 :- let X be a random variable then $F : \mathbb{R} \to [0,1]$ is defined by

$$F(x) = \mathbb{P}(X \leq x)$$

Then F is called the distribution function of X.

- One needs to understand F for both discrete and continuous random variables.