- Quiz - today - Hope you were able to login & practice submitting a quiz.

Recall :-

S - Sample space (any set) (countable) - [elements are called outcomes] listing of all possible outcomes

$\mathcal{F}$ - Events $\mathcal{F} = \mathcal{P}(S)$ temporary definition

Probability $\mathbb{P} - \mathcal{F} \longrightarrow [0,1]$ - function

Axioms

(i) $\mathbb{P}(S) = 1$

(ii) $\{E_k\}_{k \geq 1}$ $E_k \in \mathcal{F}$, $E_i \cap E_j = \phi$ $i \neq j$

$$\underbrace{\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right)}_{\in [0,1]} = \underbrace{\sum_{j=1}^{\infty} \mathbb{P}(E_j)}$$

$T_n = \sum_{j=1}^{n} \mathbb{P}(E_j)$

$\lim_{n \to \infty} T_n = \alpha$ - exists

$[0,1] \ni \alpha := \sum_{j=1}^{\infty} \mathbb{P}(E_j)$

Experiment

- choosing one outcome from a sample space

Section 1.1.2 Basic Properties

• - Probability - Definition 1.1.3

• Chance - "Relative frequency" ; $\dfrac{\text{" \# favorable outcomes "}}{\text{\# outcomes}}$

Theorem 1.1.4 : let $\mathbb{P}$ be a Probability on Sample Space S.

Then ;

① $\mathbb{P}(\phi) = 0$

② If $E_1, E_2, .. E_n$ events for some $n \geq 1$, are

(ie $E_i \cap E_j = \phi$, $i \neq j$) disjoint then

$$\mathbb{P}\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mathbb{P}(E_j)$$

③ E & F are events with $E \subseteq F$

$$\mathbb{P}(E) \leq \mathbb{P}(F)$$

④ E and F are events with $E \subseteq F$

$$\mathbb{P}(F \setminus E) = \mathbb{P}(F) - \mathbb{P}(E)$$

F/E $=$ $\{x \in S \mid x \in F \text{ and } x \in E^c\}$

⑤ let E^c be the complement of E.

Then $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

⑥ If E and F are two events then

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$

Proof :-

① $\mathbb{P}(\phi) = 0$

– Define $E_j = \phi$ for $j \geq 1$ [E_j's are disjoint sequence of events]

– Axiom ② in Definition 1.1.3 will imply

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j) \quad - (*)$$

(**) – $\phi = \bigcup_{j=1}^{\infty} E_j$ & $T_n = \sum_{j=1}^{n} \mathbb{P}(E_j) = n\,\mathbb{P}(\phi)$

From (*), (**),

$$\mathbb{P}(\phi) = \underbrace{\lim_{n\to\infty} n\,\mathbb{P}(\phi)}_{\mathbb{P}(\phi)+\mathbb{P}(\phi)+\mathbb{P}(\phi)+\cdots} \quad -(+)$$

$0 \le \mathbb{P}(\phi) \le 1$, (+), $\Rightarrow$ $\mathbb{P}(\phi) = 0$ $-(Ex)$

□

(2) $\{E_j\}_{j=1}^{n}$ $E_k \cap E_\ell = \phi$ $k \ne \ell$, $n \ge 1$

$$\mathbb{P}\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mathbb{P}(E_j)$$

$E_1, E_2, \ldots, E_n$ –, Given, $\underbrace{E_j = \phi \quad \forall j \ge n+1}_{\text{Define}}$

Consider the sequence $\{E_n\}_{n\ge 1}$. Note $E_\ell \cap E_m = \phi$ $\ell \ne m$ $\Rightarrow$ by second axiom of Definition 1.1.3

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(E_j) \quad - (+)$$

– $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{n} E_j \;\bigcup_{j=n+1}^{\infty} E_j = \bigcup_{j=1}^{n} E_j \cup \phi = \bigcup_{j=1}^{n} E_j$ – (I)

– $T_m = \sum_{j=1}^{m} \mathbb{P}(E_j)$ & $\mathbb{P}(E_k) = \mathbb{P}(\phi) = 0$ $\forall k \ge n+1$

↑ Part (1)

$\Rightarrow$ $T_m = \sum_{j=1}^{n} \mathbb{P}(E_j)$ $\forall\, m \ge n$ – (II)

$$\boxed{T_m \to \sum_{j=1}^{n} E_j \text{ as } m\to\infty}$$

(i), (ii), (iii) $\Rightarrow$

$$\mathbb{P}\left(\bigcup_{j=1}^{n} E_j\right) = \sum_{j=1}^{n} \mathbb{P}(E_j)$$

$\square$

E and F are two events with $E \subseteq F$

(3) - $\mathbb{P}(E) \le \mathbb{P}(F)$

(4) $\mathbb{P}(F \setminus E) = \mathbb{P}(F) - \mathbb{P}(E)$

Use part 2, above : $E_1 = E$, $E_2 = F \setminus E$

$\Rightarrow$ E_1, E_2 are events , $E_1 \cap E_2 = \phi$

$n=2$
E_1, E_2 $\rightarrow$ $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$

$(E \subseteq F)$ $\underbrace{\mathbb{P}(F)}$ $= \mathbb{P}(E) + \mathbb{P}(F \setminus E)$ — (6)

(4) - holds by (6)

Also (3) holds, as by (4)

$\mathbb{P}(F) - \mathbb{P}(E) = \mathbb{P}(F \setminus E) \ge 0$ by definition

$\Rightarrow \mathbb{P}(F) \ge \mathbb{P}(E)$ $\square$

(5) $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

Apply (4) $F = S$, $E = E$

$\Rightarrow \mathbb{P}(F \setminus E) = \mathbb{P}(S) - \mathbb{P}(E)$

$\Rightarrow \mathbb{P}(S \setminus E) = 1 - \mathbb{P}(E)$

$\Rightarrow \mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

(6) E and F are two events

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$

– $$E \cup F \underset{(Ex)}{=} E \cup \underbrace{F \setminus E}_{\text{Disjoint union}}$$

Apply part 2, with $E_1 = E$ & $E_2 = F \setminus E$

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F \setminus E) \quad - \quad \mathrm{I}$$

– $F \setminus E \subseteq F$ & $F \setminus (F \setminus E) \underset{(Ex)}{=} E \cap F$

Apply part (4) to $F \setminus E$, F

$$\mathbb{P}(E \cap F) = \mathbb{P}(F) - \mathbb{P}(F \setminus E)$$

Rearrange

$$\mathbb{P}(F \setminus E) = \mathbb{P}(F) - \mathbb{P}(E \cap F) \quad \mathrm{II}$$

I and II $\quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

$\square$

Example 1.1.7. Suppose we know that there is a 60% chance that it will rain tomorrow and a 70% chance that high temperature will be above

30°C. Suppose we also know that there is a 40% chance that high temperature will be above 30°C and it will rain.

How likely is it tomorrow will be a dry day and that temperature does not go above 30°C?

Answer :-

$E = \{\text{rains tomorrow}\}$

$F = \{\text{high temperature exceeds } 30°C \text{ tomorrow}\}$

$\mathbb{P}(E) = 0.6$, $\mathbb{P}(F) = 0.7$, $\mathbb{P}(E \cap F) = 0.4$

Need to find : $\mathbb{P}(E^c \cap F^c)$

$$E^c \cap F^c = (E \cup F)^c$$

From Theorem 1.1.4 (6) and (5)

$$\mathbb{P}((E \cup F)^c) \underset{(5)}{=} 1 - \mathbb{P}(E \cup F)$$

$$\underset{(6)}{=} 1 - [\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)]$$

$$= 1 - [0.6 + 0.7 - 0.4] = 0.1$$

$\square$

ICING on the Cake Questions :-

ICING-Q1 $\{A_n\}_{n\geq 1}$ – sequence of events } increasing sequence of events

$$A_n \subseteq A_{n+1} \quad \forall n \geq 1$$

$$\Rightarrow \quad \mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$$

Remarks :-

Theorem 1.1.4 ③

- $\mathbb{P}(A_n) \leq \mathbb{P}(A_{n+1}) \quad \forall n \geq 1$
- $\bigcup_{j=1}^{n} A_j = A_n$ Ex

ICING-Q2 $\{A_n\}_{n\geq 1}$ is a sequence of decreasing events

$A_n \supseteq A_{n+1} \quad \forall n \geq 2$ then

$$\mathbb{P}\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$$