Recall :-

- $S = \{\text{Success, failure}\}$ $\mathcal{F} = \mathcal{P}(S)$ } Bernoulli(p)
  $\mathbb{P}: \mathcal{F} \to [0,1]$ $\mathbb{P}(\{\text{Success}\}) = p$; $0 \le p \le 1$

- n - independent Bernoulli(p) trials -

(a) What is the probability of observing k successes?

$S = \{0, 1, 2, \dots, n\}$ $\mathcal{F} = \mathcal{P}(S)$

$\mathbb{P}: \mathcal{F} \to [0,1]$ $\mathbb{P}(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k}$ } Binomial(n, p)

(b) What is the most likely number of successes?

Mode :- $k = \lfloor p(n+1) \rfloor$ | $p(n+1) \in \mathbb{Z}$ $k = p(n+1)$ and $k+1$ were :
$\mathbb{P}(\{k\}) = \mathbb{P}(\{k+1\})$

(c) How many attempts must be made before the first success is observed? [Perform Bernoulli(p) trials till you observe a success]

Geometric (p) : $S = \{1, 2, 3, \dots\}$ $\mathcal{F} = \mathbb{N}$, $\mathcal{F} = \mathcal{P}(S)$

$\mathbb{P}: \mathcal{F} \to [0,1]$ $\mathbb{P}(\{k\}) = (1-p)^{k-1} p$, $k \in \mathbb{N}$

— Cleanup - Making Part (c) precise

[Complete] Definition of $\mathbb{P}: \mathcal{F} \to [0,1]$

- $\mathbb{P}(\{k\}) = (1-p)^{k-1} p$ $k \in \mathbb{N}$, $0 \le p \le 1$

$A \in \mathcal{F}$ $\mathbb{P}(A) = \sum_{k \in A} \mathbb{P}(\{k\})$

Verify: $\mathbb{P}$ is indeed a probability

- $\mathbb{P}(\{k\}) \geq 0$ $\forall k \geq 1$ $\boxed{\text{\& } \mathbb{P}(\{k\}) > 0 \text{ if } 0 < p < 1 \ \forall k \geq 1}$

• $A \subseteq B \subseteq S$; $\quad \mathbb{P}(A) = \sum_{k \in A} \mathbb{P}(\{k\}) \leq \sum_{k \in B} \mathbb{P}(\{k\}) = \mathbb{P}(B)$

Analysis I

• $|A| \leq |B| < \infty$ — ✓

• $|B| = \infty$, B – countably infinite

Ex – to show

$\Rightarrow \quad A \subseteq S$

$\mathbb{P}(A) \leq \mathbb{P}(S)$

If we show $\mathbb{P}(S) = 1$ then $\mathbb{P} : \mathcal{F} \to [0,1]$ & satisfies axiom 1 of definition of Prob.

— $\mathbb{P}(S) = \sum_{k \in S} \mathbb{P}(\{k\}) = \sum_{k=1}^{\infty} \mathbb{P}(\{k\}) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$

$= 1.$

Analysis I

$$T_n = \sum_{k=1}^{n} (1-p)^{k-1} p \ , \quad n \geq 1$$

$$= p \, \frac{1-(1-p)^n}{1-(1-p)} \quad \text{[Induction Ex]}$$

$$= 1 - (1-p)^n$$

$T_n \to 1$ as $n \to \infty$

Ex: $0 < 1-p < 1$ then $(1-p)^n \to 0$ as $n \to \infty$

For Axiom 2

$\{E_k\}_{k \geq 1}$ $\quad E_k$ are disjoint events and we need to show

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

Analysis I

Ex: S – Countable

So

$$\left.\begin{aligned} \mathbb{P}(E_k) &:= \sum_{m \in E_k} \mathbb{P}(\{m\}) \quad \forall k \geq 1 \\ \mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) &:= \sum_{m \in \bigcup_{k=1}^{\infty} E_k} \mathbb{P}(\{m\}) \end{aligned}\right\} \Rightarrow \sum_{k=1}^{\infty} \mathbb{P}(E_k) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right)$$

ICING on the cake (d) :- On an average how many successes will there be ?

Question

Ex:- How would be define a precise notion of average ?

[Answer this when we do chapter 4]

Install R on your Computer / device [1-QUIZ - mock] / [HW 1]

- [NOT part of syllabus of the course] — Book - section 1.5

(R) - Ex1 - Use R as a calculator

(R) - Ex2 → To calculate Binomial (n, p) probabilities

Example $p = 0.35928$

$n = 1000$ $k = 565$

? $P(\{k\}) = \binom{1000}{k} (0.35928)^k (1 - 0.35928)^{1000-k}$

(R) - Ex3 :- Computing Binomial (n, p) and Geometric (p) Probabilities

- dbinom(_ , _ , _) → k, n, p

- dgeom(_ , _) → k, p

2.2 Poisson Approximation

As we have seen calculating Binomial probabilities may be a challenge when n - large.

Example 2.2.1 :- A small college has 1460 students. Assume that birth rates are constant through out the year & each year has 365 days. What is the probability that five or more students were born on January 1st (new year day)?

Ex:- Ans: $1 - \sum_{k=0}^{4} \binom{1460}{k} \left(\frac{1}{365}\right)^{k} \left(\frac{364}{365}\right)^{1460-k}$

- Computing this answer by hand is hard & is a calculator is tedious.

Q: Is there an approximate answer that is easy to obtain?

Answer: [One such approximation] $p \lll$, $n \ggg$ such that $np \equiv$ remains a constant [YES]

[i.e. $p \equiv p(n)$ $np \longrightarrow \lambda$ as $n \to \infty$]

— Poisson approximation.

$$\binom{n}{k} p^{k} (1-p)^{n-k} \approx \ldots ?$$

Theorem 2.2.2:- let $\lambda > 0$; $k \geq 1$; $n \geq \lambda$, $n \in \mathbb{N}$; and $p = \frac{\lambda}{n}$

$A_k = \{ k$ - success in n Bernoulli (p) trials$\}$

it then follows that

$$\lim_{n \to \infty} \mathbb{P}(A_k^{(n)}) = \frac{e^{-\lambda} \lambda^k}{k!}$$

Observations - Analysis I preparation

(definition $e \in \mathbb{R}$ as Ex) ; $e^{-\lambda} \equiv ?$

- $e = ?$

$e^{-\lambda} \equiv ?$

- $A_k^{(n)} = \{ k$ - success in n Bernoulli $(\frac{\lambda}{n})$ trials$\}$

$\mathbb{P}(A_k) \equiv \mathbb{P}(A_k^{(n)})$

Fix $k \geq 1$

Trying to understand: $\lim_{n \to \infty} \mathbb{P}(A_k^{(n)})$ - for fixed $k \geq 1$

- Ux: $\frac{e^{-\lambda} \lambda^k}{k!}$ serves as an approximate answer for $\mathbb{P}(A_k^{(n)})$ for $n \ggg$; $p = \frac{\lambda}{n}$.

Proof:- Fix $k \geq 1$, $\lambda > 0$, let $n > \lambda$, $p = \frac{\lambda}{n}$.

$$\mathbb{P}(A_k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbb{P}(A_k^{(n)}) = \frac{n!}{k!\,(n-k)!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}\,\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

Compute: $\lim_{n\to\infty} \mathbb{P}(A_k^{(n)})$ ..

$$\mathbb{P}(A_k^n) = \frac{\lambda^k}{k!}\,\frac{n(n-1)\cdots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!}\,\underbrace{1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{(k-1)}{n}\right)}_{\text{Product of } k \text{ terms}}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

Analysis I facts :- Ex.

1. $\lim_{n\to\infty}\left(1-\frac{r}{n}\right) = 1$ for $r\geq 1$;
2. $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} = 1$ $\forall\, \lambda>0,\ k\geq 1$; and
3. $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$ for all $\lambda\geq 0$
4. $a_n \to a$ as $n\to\infty$; $b_n\to b$ as $n\to\infty$
   $a_n b_n \longrightarrow ab$ as $n\to\infty$

$$\mathbb{P}(A_k^{(n)}) = \frac{\lambda^k}{k!} \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) \cdot \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

as $n \to \infty$

(analysis 1) ① & ④ ③ ②

$$\underbrace{\frac{\lambda^k}{k!} \cdot 1 \cdot e^{-\lambda} \cdot 1}_{④}$$

$$\lim_{n\to\infty} \mathbb{P}(A_k^{(n)}) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad \square$$

- Poisson (λ) Distribution:

$S = \{0\} \cup \mathbb{N}$; $\mathcal{F} = \mathcal{P}(S)$ $\quad \mathbb{P} : \mathcal{F} \to [0,1]$

$k \in S$; $\quad \mathbb{P}(\{k\}) = \dfrac{e^{-\lambda}\lambda^k}{k!}$

- $\mathbb{P}$ – Probability on S is called Poisson distribution on S. $\square$