Recall :- Sampling and repeated trials

- Bernoulli trials $\quad S = \{ \text{success}, \text{failure} \}$

Bernoulli (p) distribution — $\begin{cases} \mathcal{F} = \mathcal{P}(S) \\ \mathbb{P} : \mathcal{F} \to [0,1] \\ \mathbb{P}(\{\text{success}\}) = p \end{cases}$; $0 \le p \le 1$

Example 2.1.1. Roll a die two times.

Q:- What is the chance that we have exactly one 6 in two rolls?

- Each roll — Bernoulli(1/6) ⟵ Treated occurrence of 6 as a success.

- $\mathbb{P}$ (Exactly one 6 in two rolls)

model as Bernoulli (1/6) trials ⟵ $= \mathbb{P}(\{\text{success}, \text{failure}\} \cup \{\text{failure}, \text{success}\})$

mutually exclusive ⟵ $= \mathbb{P}(\{\text{success}, \text{failure}\}) + \mathbb{P}(\{\text{failure}, \text{success}\})$

independent trials ⟵ $= \mathbb{P}(\{\text{success}\})\, \mathbb{P}(\{\text{failure}\}) + \mathbb{P}(\{\text{failure}\})\, \mathbb{P}(\{\text{success}\})$

Bernoulli (1/6) ⟵ $= \frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{1}{6}$

$= \frac{10}{36}$ □

Fact:- $a, b \in \mathbb{R}$; $n \geq 1$;

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof by [Induction]

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Probability — Bernoulli (p) trials n - independent — $a = p$, $b = 1-p$ $\quad 0 \le p \le 1$

Fact: $1 = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \quad \forall\, n \geq 1$

Example 2.1.2 :- let $n \geq 1$ be given. After performing n independent Bernoulli (p) trials we are typically interested in

(a) What is the probability k- successes ?

$$S = \{ (\omega_1, \omega_2, \dots, \omega_n) \mid \omega_i \equiv \text{success or failure} \} \quad |S| < \infty$$
$$1 \leq i \leq n$$

$$\mathcal{F} = \mathcal{P}(S).$$

$$\mathbb{P} : \mathcal{F} \to [0,1]$$

Each trial is independent

$\mathbb{P}(\{\text{success}\}) = p$

$\mathbb{P}(\{\text{failure}\}) = (1-p)$

$$\mathbb{P}(\{\omega_1, \dots \omega_n\}) = \prod_{i=1}^{n} \mathbb{P}(\{\omega_i\})$$

$$= p^{\#\{i : \omega_i = \text{success}\}} (1-p)^{\#\{i : \omega_i = \text{failure}\}} \quad \text{—} (*)$$

Q :- $B_k = \{ k \text{ successes in } n \text{ trials} \}$; $\mathbb{P}(B_k) = ?$

A :- $$\mathbb{P}(B_k) = \sum_{\omega \in B_k} \mathbb{P}(\{\omega\}) \qquad \omega = (\omega_1, \dots, \omega_n)$$

$$\underset{(\omega_1, \dots, \omega_n)}{\overset{\shortparallel}{\omega}} \in B_k \implies \#\{i : \omega_i = \text{success}\} = k$$
$$\& \ \#\{i : \omega_i = \text{failure}\} = n-k$$

$$\therefore \omega \in B_k \implies \mathbb{P}(\{\omega\}) = p^k (1-p)^{n-k}$$

$$\therefore \mathbb{P}(B_k) = \sum_{\omega \in B_k} p^k (1-p)^{n-k}$$

$$= \quad |B_k| \; p^k (1-p)^{n-k}$$

Now, $|B_k| \equiv$ number of ways k successes can occur in n trials [- choose k trials for success among n trials]

$$\equiv \binom{n}{k}$$

$$\therefore \quad \mathbb{P}(B_k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le k \le n \quad \square$$

As in Question: n-trials & and our only interest is in the number of successes. Then

$T = \{0, 1, 2, \ldots n\}$ $\qquad \mathcal{F} = \mathcal{P}(T)$

$\mathbb{P} : T \to [0,1]$

$$\mathbb{P}(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k} \qquad 0 \le k \le n$$

$$\left[\mathbb{P}(T) = \sum_{k=0}^{n} \mathbb{P}(\{k\}) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1 \right]$$

— Binomial(n, p) ; $T = \{0, 1, 2 \ldots n\}$; $\mathbb{P}(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \le k \le n$.
$0 \le p \le 1$

Example 2.1.2 :- (Contd.) let $n \ge 1$ be given. A-ree performing n independent Bernoulli (p) trials ; we are typically interested in another question:-

(b) What is the most likely number of successes?

[Mode of Binomial (n,p)] :- $p=0$ or $p=1$ [Trivial]

Ans: $k=0$; $k=n$

$0<p<1$:- $B_k = \{k \text{ success in } n \text{ trials}\}$

Q: Find k such that $\mathbb{P}(B_k)$ is as large as possible.

A: For, $0 \le k \le n-1$

$$\frac{\mathbb{P}(B_{k+1})}{\mathbb{P}(B_k)} \overset{(a)}{=} \frac{\binom{n}{k+1} \cancel{p}^{k+1} \cancel{(1-p)}^{n-(k+1)}}{\binom{n}{k} \cancel{p}^{k} \cancel{(1-p)}^{n-k}} = \frac{\frac{\cancel{n!}}{(k+1)!\,(n-(k+1))!}}{\frac{\cancel{n!}}{k!\,(n-k)!}} \frac{p}{1-p}$$

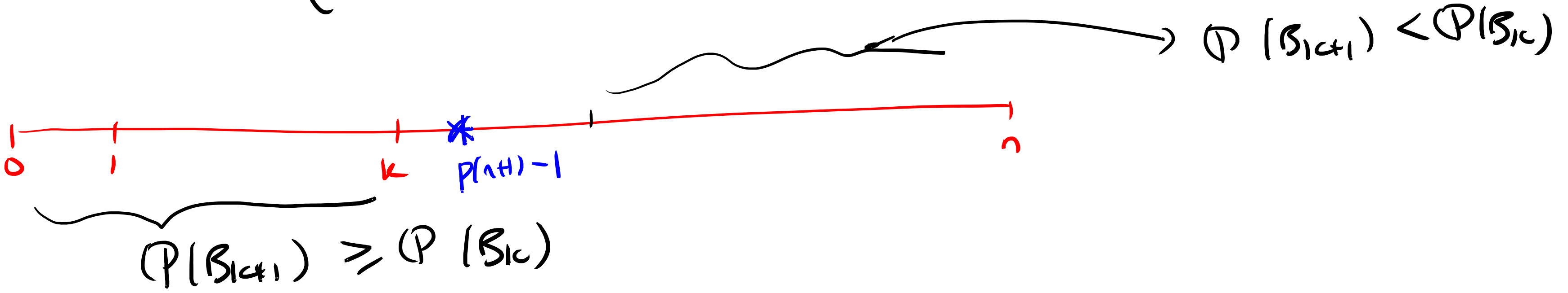
$$= \frac{p}{1-p} \, \frac{n-k}{k+1}$$

Cases :- $0 \le k \le n-1$

- $\mathbb{P}(B_{k+1}) \geq \mathbb{P}(B_k) \iff \frac{\mathbb{P}(B_{k+1})}{\mathbb{P}(B_k)} \geq 1 \iff \frac{p}{1-p}\,\frac{n-k}{k+1} \geq 1$

$$\iff pn - pk \geq (1-p)(k+1)$$

$$\iff k \le p(n+1) - 1 \quad \text{—} (*)$$



In other words if k starts at 0 and begins to increase the probability of achieving k - success will increase while $k < p(n+1) - 1$ and then will decrease once $k > p(n+1) - 1$.

Ans: The most likely number of success is the integer value of k for which

$$k-1 \le p(n+1) - 1 < k$$

$$\Rightarrow \quad k = \lfloor p(n+1) \rfloor$$

Unusual case: $p(n+1)$ - integer :- ; $k = p(n+1)$

$$\Rightarrow \quad \frac{\mathbb{P}(B_{k+1})}{\mathbb{P}(B_k)} = 1$$

ie {k success} and {k+1 success} have the same probability. Answer is shared by $k+1$, k with $k = p(n+1)$

Example 2.1.2 :- (Contd). let $n \geq 1$ be given. After performing n independent Bernoulli(p) trials we are typically interested in another question:-

(c) How many attempts must be made before the first success is observed?

Ans:- It's possible that ; first success occurs in 1^{st} trial

Then :- $\mathbb{P}$(Success in 1^{st} trial) $= p$

; first success occurs in 2^{nd} trial

Then : $\mathbb{P}$(first success in 2^{nd} trial) $= \mathbb{P}(\{\text{failure, success}\})$

$= \mathbb{P}(\{\text{failure}\})\ \mathbb{P}(\{\text{success}\})$

$= (1-p)\ p$

the k^{th} trial} $k \geq 3$

$C_k = \{$ the first success occurs on the k^{th} trial $\}$

$1 \leq i \leq k$, $A_i = \{$ i^{th} trial is a success $\}$, $\mathbb{P}(A_i) = p$ and $\mathbb{P}(A_i^c) = 1-p$

$$C_k = A_1^c \cap A_2^c \cdots \cap A_{k-1}^c \cap A_k$$

$$\mathbb{P}(C_k) = \mathbb{P}(A_1^c \cap A_2^c \cdots \cap A_{k-1}^c \cap A_k)$$

independence $\leftarrow$ $= \prod_{i=1}^{k-1} \mathbb{P}(A_i^c)\ \mathbb{P}(A_k)$

$= (1-p)^{k-1}\ p$ $\quad k \geq 1$ $\quad \square$

Geometric (p) - distribution : $S = \{1, 2, 3, \ldots\} = \mathbb{N}$, $\mathcal{F} = \mathcal{P}(S)$

$$\mathbb{P}(\{k\}) = (1-p)^{k-1}\ p. \qquad k \geq 1$$

Ex :- $\sum_{k=1}^{\infty} \mathbb{P}(\{k\}) = 1$ $\quad \square$