Recall :- Independence

- Occurrence of one event does not affect the probability of the other.

Example 1.4.1 $S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$

$C = \{hhh, hht, hth, htt\}$, $D = \{hhh, hht, thh, tht\}$

$\mathbb{P}(C) = \frac{1}{2}$, $\mathbb{P}(D) = \frac{1}{2}$; $\mathbb{P}(C|D) = \mathbb{P}(C|D^c) = \frac{1}{2}$

Observation: • $\mathbb{P}(C) = \mathbb{P}(C|D) = \mathbb{P}(C|D^c)$ — (*)

occurrence or non-occurrence of D had no effect on probability of C.

• (*) you needed $\mathbb{P}(D) > 0$ and $\mathbb{P}(D^c) > 0$

rewrite (*) $\mathbb{P}(C \cap D) = \mathbb{P}(C)\,\mathbb{P}(D)$

Definition 1.4.2 [Independence] Two events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Example 1.4.3 :- Roll a dice two times. [Equally likely outcome]

$|S| = 36$

$E = \{6 \text{ appears in the 1st roll}\} = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$F = \{6 \text{ appears in the 2nd roll}\} = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\}$

$E \cap F = \{(6,6)\}$

$$\mathbb{P}(E) = \frac{6}{36} = \mathbb{P}(F)$$

$$\mathbb{P}(E \cap F) = \frac{1}{36}$$

$$\therefore \mathbb{P}(E \cap F) = \mathbb{P}(E) \cdot \mathbb{P}(F)$$

$\therefore$ E & F are independent $\square$

Observations :-

- A and B are independent events $\iff \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ [Implied by 4 equalities]

Ex :- $0 < P(B) < 1$, $0 < \mathbb{P}(A) < 1$

[Roles A & B can be reversed]

$$\iff \mathbb{P}(A \mid B^c) = \mathbb{P}(A)$$
$$\iff \mathbb{P}(A^c \mid B) = \mathbb{P}(A^c)$$
$$\iff \mathbb{P}(A^c \mid B^c) = \mathbb{P}(A^c)$$
$$\iff \mathbb{P}(B^c \mid A^c) = \mathbb{P}(B^c)$$
$$\iff \mathbb{P}(B \mid A^c) = \mathbb{P}(B)$$

- A_1, A_2, A_3 — to say they are independent if

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) \quad -(**)$$

(**) is NOT Enough to quantify 3 events are independent.

only Require A_i, A_j are independent for $i \neq j$ [NOT Enough]

$S = \{hh, ht, th, tt\}$

$A_1 = \{hh, tt\}$; $A_2 = \{hh, ht\}$

$A_3 = \{hh, th\}$

- $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{1}{2}$

- $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_3 \cap A_1) = \frac{1}{4}$

$\mathbb{P}(A_1 \cap A_2 \cap A_3) = 1/4$

$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ $\quad 1 \le i \neq j \le 3$

But $\mathbb{P}(A_1 \cap A_2 \cap A_3) \neq \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$

(**) $\Rightarrow$ pairwise independence

(FALSE)

Ex: - in next H.w.

Needed something else

(**) with A_i and A_i^c at all positions i.

Definition 1.4.5 (Mutual independence)

A finite collection of events $A_1, A_2, \ldots, A_n$ is mutually independent if

$$\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \prod_{i=1}^{n} \mathbb{P}(E_i) \quad (\ddagger)$$

where $E_i = A_i$ or $E_i = A_i^c$

Observations

($\ddagger$) $\Leftarrow$ to the same definition in $n=2$ case

and if $n>2$ case it also is consistent.

Ex: If $A_1, A_2, \ldots A_n$ are independent for $n \geq 1$ then

($\ddagger$) $A_{i_1}, \ldots A_{i_k}$ are also independent with $k<n$ and $1 \le i_1, i_2, \ldots i_k \le n$ $\quad i_l \neq i_m$

- $\{A_t : t \in I\}$ for some index set I. Then we say the collection is mutually independent if any finite sub-collection is mutually independent [Definition 1.4.7]

$\mathcal{F}$ $\quad \mathbb{P}(E_1 \cap E_2, \ldots \cap E_n) = \prod_{i=1}^{n} \mathbb{P}(E_i) \Big\} \equiv 2^n$ equations

$E_i = A_i$ or A_i^c

2. Sampling & Repeated Trials

$(S, \mathcal{F}, \mathbb{P})$: $S \supseteq A$ - event of interest

Success - if A occurs in an experiment

Failure - A does NOT occur

Experiment	S	Event - A	$\mathbb{P}(A)$
- Toss a fair coin	{Head, Tail}	{H}	$\frac{1}{2}$
- Roll a die	{1, 2, 3, 4, 5, 6}	{2, 4}	$\frac{1}{3}$

Application: repeat the experiment many times (independently) & you would be interested in the number of successes. [≡ Sampling from a large population]

2.1 Bernoulli Trials [James Bernoulli = 17^{th} Century]

- Mathematical framework for independent trials of an experiment.
- Designate outcomes < Success / Failure
- $p \equiv \mathbb{P}(\{\text{success}\})$ – at each trial.

Bernoulli (p) – 1-trial $S = \{\text{success, failure}\}$

$$\mathbb{P}(\{\text{success}\}) = p.$$

$$\mathcal{F} = \{\phi, \{\text{success}\}, \{\text{failure}\}, S\} \qquad \mathbb{P} : \mathcal{F} \to [0,1]$$

$\mathbb{P}(\phi) = 0$, p, $\mathbb{P}(\{\text{failure}\}) = 1 - p$, 1

- Often we are interested in performing multiple Bernoulli (p) trials.

Example 2.1.1.– Suppose we roll a dice two times.

Q: How likely is it that we observe exactly one 6 in the two rolls?

Method I

Recall :– Example 1.4.3

$$S = \left\{ \begin{matrix} (1,1) & \cdots & (1,6) \\ (6,1) & \cdots & (6,6) \end{matrix} \right\} \quad |S| = 36$$

$$\text{Event} = \left\{ \begin{matrix} (1,6) & (2,6) & ,(3,6) & ,(4,6) & ,(5,6) \\ (6,1) & (6,2) & (6,3) & ,(6,4) & (6,5) \end{matrix} \right\} \quad \therefore \mathbb{P}(E) = \frac{|E|}{|S|} = \frac{10}{36}$$

$\boxed{\text{Method II}}$

Designate $\equiv \begin{cases} \text{6 occurring in a roll} \equiv \text{Success} \\ \text{6 not occurring in a roll} \equiv \text{Failure} \end{cases}$

$A = \{6 \text{ occurs in a roll}\} \Rightarrow \mathbb{P}(A) = 1/6$

$\mathbb{P}(\{\text{Failure}\}) = 5/6$

$A_1 = \{6 \text{ occurs in } 1^{st} \text{ roll}\}$ $\quad \mathbb{P}(A_1) = 1/6$

$A_2 = \{6 \text{ occurs in } 2^{nd} \text{ roll}\}$ $\quad \mathbb{P}(A_2) = 1/6$, $\mathbb{P}(A_2^c) = 5/6$

$$\mathbb{P}(\{\text{Success}, \text{Success}\}) = \mathbb{P}(A_1 \cap A_2)$$
$$\overset{\text{independent}}{=} \mathbb{P}(A_1)\,\mathbb{P}(A_2)$$
$$= 1/36$$

$$\mathbb{P}(\{\text{Success}, \text{Failure}\}) = \mathbb{P}(A_1 \cap A_2^c)$$
$$\overset{\text{independent}}{=} \mathbb{P}(A_1)\,\mathbb{P}(A_2^c)$$
$$= 1/6 \cdot 5/6 = \frac{5}{36}$$

$$\mathbb{P}(\{\text{Failure}, \text{Success}\}) = \mathbb{P}(A_1^c \cap A_2)$$
$$\overset{\text{independent}}{=} \mathbb{P}(A_1^c)\,\mathbb{P}(A_2)$$
$$= 5/6 \cdot 1/6 = 5/36$$

$$\mathbb{P}(\{\text{Failure}, \text{Failure}\}) = \mathbb{P}(A_1^c \cap A_2^c)$$
$$= \mathbb{P}(A_1^c)\,\mathbb{P}(A_2^c)$$
$$= \frac{25}{36}$$

$$S' = \{\{\text{Success}, \text{Success}\}, \{\text{Success}, \text{Failure}\}, \{\text{Failure}, \text{Success}\}, \{\text{Failure}, \text{Failure}\}\}$$

$$\mathcal{F} = \mathcal{P}(S')$$

$$\mathbb{P} : \mathcal{F} \to [0,1]$$

- $\mathbb{P}(\{\text{Success}, \text{Success}\}) = 1/36$
- $\mathbb{P}(\{\text{Success}, \text{Failure}\}) = 5/36$
- $\mathbb{P}(\{\text{Failure}, \text{Success}\}) = 5/36$
- $\mathbb{P}(\{\text{Failure}, \text{Failure}\}) = 1/36$

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\})$$

$$\mathbb{P}(\text{only one 6 occurs}) = \mathbb{P}(\{\text{Success}, \text{Failure}\} \cup \{\text{Failure}, \text{Success}\})$$
$$= \mathbb{P}(\{\text{Success}, \text{Failure}\}) + \mathbb{P}(\{\text{Failure}, \text{Success}\})$$
$$= 5/36 + 5/36 = 10/36 \qquad \square$$