Recall :-

- $(S, \mathcal{F}, \mathbb{P})$ $\quad A, B \in \mathcal{F} \quad \mathbb{P}(B) > 0$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

[Conditional Probability of A given B]

Example 1.3.1 $\quad S = \{hhh, hht, \dots, ttt\}$

$A = \{$ there are two or more heads $\}$

$B = \{$ there is a head in 1st toss $\}$

$$\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$$

$\mathbb{P}(A|B) = 3/4$ – different from unconditional probability

Theorem 1.3.5 : A - event $\quad \{B_j\}_{j=1}^{n}$ disjoint collection of events, $\mathbb{P}(B_i) > 0$; $A \subseteq \bigcup_{i=1}^{n} B_i$

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(B_i)\, \mathbb{P}(A|B_i) \quad - (*)$$

Ex – $n = \infty$ – $(*)$ is also true

Example 1.3.6 :-

Recall :-

quantity Table

	B1	B2	B3
Red	4	3	3
Green	3	3	4
Blue	5	2	3

- choose a Box and choose a ball in that Box

- $\mathbb{P}(R) = \cdots \underset{\text{Theorem 1.3.5}}{=}$

$$= \sum_{i=1}^{3} \mathbb{P}(B_i)\,\mathbb{P}(R|B_i)$$

$$= \cdots = \frac{121}{360}$$

- $\mathbb{P}(B_3|R) = \dfrac{\mathbb{P}(B_3 \cap R)}{\mathbb{P}(R)} = \dfrac{\mathbb{P}(R|B_3)\,\mathbb{P}(B_3)}{\mathbb{P}(R)}$

$$= \frac{(3/10)\cdot\frac{1}{3}}{121/360} = \frac{36}{121}$$

The above was part of a general framework

Theorem (Bayes) 1.3.11 Suppose A is an event $\{B_i : 1 \le i \le n\}$ - disjoint events $A \subseteq \bigcup_{i=1}^{n} B_i$

$\mathbb{P}(A) > 0$, $\mathbb{P}(B_i) > 0$ $1 \le i \le n$. Then

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}{\sum_{j=1}^{n} \mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)} \quad - (*)$$

Ex: $n = \infty$ $(*)$ holds.

We have seen so far how conditioning on an event B or events $\{B_i\}_{i=1}^{n}$, we can get the probability of another event A.

Theorem 1.3.8 $n \geq 2$. $A_1, A_2, \ldots A_n$ are a collection of events. Assume $\mathbb{P}\left(\bigcap_{j=1}^{n-1} A_j\right) > 0$.

$$\mathbb{P}\left(\bigcap_{j=1}^{n} A_j\right) = \mathbb{P}(A_1) \prod_{j=2}^{n} \mathbb{P}\left(A_j \,\middle|\, \bigcap_{k=1}^{j-1} A_k\right) \quad - (*)$$

Proof:- Ex. - general case

$\underline{n=2}$ $\quad \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 \mid A_1)$

$n=3$ $\quad \mathbb{P}(A_1 \cap \underbrace{A_2 \cap A_3}) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \cap A_3 \mid A_1)$

[Verify] $\quad = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1) \cdot \mathbb{P}(A_3 \mid A_1 \cap A_2)$

$\square$

Example 1.3.9. A probability class has 15 students

4 – are interested in hockey

8 – are interested in kabaddi

3 – are interested in kho-kho

Three different students are selected to present H.W problems. What is the probability the selection will be that the student likes kabaddi, second likes kho-kho, third student likes kabaddi? (in that order)

Answer:-

$A_1 = \{$ first selection is from kabbadi group $\}$

$A_2 = \{$ Second selection is from kho-kho group $\}$

$A_3 = \{$ third selection is from kabbadi group $\}$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = ?$$

$$= \mathbb{P}(A_1)\ \mathbb{P}(A_2 | A_1)\ \mathbb{P}(A_3 | A_1 \cap A_2)$$

Theorem 1.3.8

$$\mathbb{P}(A_1) = \frac{8}{4+8+3} = \frac{8}{15} \ ; \ \mathbb{P}(A_2 | A_1) = \frac{3}{14} \ ; \text{ and}$$

$$\mathbb{P}(A_3 | A_1 \cap A_2) = \frac{7}{13} \implies \mathbb{P}(A_1 \cap A_2 \cap A_3) = \frac{8 \cdot 3 \cdot 7}{15 \cdot 14 \cdot 13} = 4/65 \ \square$$

formula illustrates

– (*); the fact that conditioning on an event may be viewed as modifying the sample space and the probability on it

Example 1.3.12

Shyam – tested positive for flu.

Given: 95% of people with flu test positive

but 2% of people without flu also test positive.

Known: 1% of the population has the flu.

Q: What is probability that Shyam had the flu given that he tested positive?

Ans:

$A = \{\text{Shyam has flu}\}$

$B = \{\text{Shyam test positive}\}$

$P(B|A) = 0.95$; $P(B|A^c) = 0.02$

$P(A) = 0.01$

$$P(A|B) = ? \overset{\text{Bayes}}{\underset{\text{Theorem}}{=}} \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)}$$

to $B_1 = A$
$B_2 = A^c$

$$= \frac{(0.95)(0.01)}{(0.95)(0.01) + (0.02)(1-0.01)} = 0.324.$$

D

1.4 Independence

Example:- Suppose we toss a coin three times.

$$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$$

Done Earlier :-

$A = \{$ there are two or more heads in 3 tosses $\}$

$B = \{$ first toss is a head $\}$-

$$\mathbb{P}(A) = \frac{1}{2}, \quad \mathbb{P}(B) = \frac{1}{2} \qquad \mathbb{P}(A|B) = 3/4$$

- The above was an instance were the occurrence of one event affected the Probability of occurrence of another.

$C = \{\text{first toss is a head}\} = \{hhh, hht, hth, htt\}$

$D = \{\text{second toss is a head}\} = \{hhh, hht, tht, thh\}$

$\mathbb{P}(C) = \frac{|C|}{|S|} = \frac{4}{8} = \frac{1}{2}$; $C \cap D = \{hhh, hht\}$

$\mathbb{P}(D) = \frac{|D|}{|S|} = \frac{4}{8} = \frac{1}{2}$

$C \cap D^c = \{hth, htt\}$

$\mathbb{P}(C \cap D) = \frac{|C \cap D|}{|S|} = \frac{2}{8} = 1/4$

Two Computations:

$$\mathbb{P}(C|D) = \frac{\mathbb{P}(C \cap D)}{\mathbb{P}(D)} = \frac{1/4}{1/2} = 1/2 \equiv \mathbb{P}(C)$$

$$\mathbb{P}(C|D^c) = \frac{\mathbb{P}(C \cap D^c)}{\mathbb{P}(D^c)} = \frac{2/8}{1 - 1/2} = \frac{1}{2} \equiv \mathbb{P}(C)$$

Occurrence or Non-occurrence of D — Does NOT affect the probability of occurrence of C.

– $\mathbb{P}(C|D) = \mathbb{P}(C) \iff \frac{\mathbb{P}(C \cap D)}{\mathbb{P}(D)} = \mathbb{P}(C) \iff \mathbb{P}(C \cap D) = \mathbb{P}(C)\mathbb{P}(D)$

$\mathbb{P}(D) > 0$

Definition 1.4.2 [Independence] Two events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) \quad - \circledast$$

Suppose we have 3 three events A_1, A_2, A_3.

Q: When do we say three events are independent?

Naive Answer :- $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3) \quad - \text{(I)}$

[need more]

Example 1.4.4 Suppose we toss a fair coin 2 times.

$A_1 = \{hh, tt\}$, $A_2 = \{hh, ht\}$, $A_3 = \{hh, th\}$

$$\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{2}{4} = \frac{1}{2}$$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3) = 1/4$$

$$\Rightarrow \quad \mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\,\mathbb{P}(A_j) \qquad 1 \le i \ne j \le 3$$

$\Rightarrow$ A_i & A_j are independent, $i \ne j$.

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(\{hh\}) = 1/4$$

$\Rightarrow \quad \mathbb{P}(A_1 \cap A_2 \cap A_3) \neq \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$

D

- Its tempting $(\ddagger)$ as definition of independence for 3 events (to mirror $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$ for A & B independent)

but 3 events being independent should imply they are pairwise independent. The above example shows pairwise independence $\not\Rightarrow$ $(\ddagger)$

– [Need more ?]

$(*)$ $\quad \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

$\Updownarrow \quad \mathbb{P}(A \cap B^c) = \mathbb{P}(A)\,\mathbb{P}(B^c)$

<u>Ex</u>: $\Updownarrow \quad \mathbb{P}(A^c \cap B) = \mathbb{P}(A^c)\,\mathbb{P}(B)$

$\Updownarrow \quad \mathbb{P}(A^c \cap B^c) = \mathbb{P}(A^c)\,\mathbb{P}(B^c)$

<u>Definition 1.4.5</u> [Mutual Independence] A finite collection of n-events $A_1, A_2, .., A_n$ is mutually independent

If $\mathbb{P}(E_1 \cap E_2 \cap .. E_n) = \mathbb{P}(E_1)\,\mathbb{P}(E_2) \,..\, \mathbb{P}(E_n)$ — [2^n equations]

where $E_i = A_i$ or A_i^C.

An arbitrary collection of events $\{A_t : t \in I\}$ for some index set I is mutually independent if every finite subcollection is mutually independent.