Recall :- S - sample space $\quad \mathbb{P}$ - Probability

$\mathcal{F}$ - event space

A, B are two events. $\mathbb{P}(B) > 0$

Probability of A given occurrence of B

denoted by $\mathbb{P}(A \mid B) := \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ $\leftarrow$ Applies any sample space $(S, \mathcal{F}, \mathbb{P})$

Motivation

$|S| < \infty$, $\quad \mathbb{P}(\{\omega\}) = \dfrac{1}{|S|} \quad \forall\, \omega \in S$

A, B are two events $|A|, |B| \neq 0$ $\equiv$ $\mathbb{P}(A \mid B) := \dfrac{|A \cap B|}{|B|} = \dfrac{|A \cap B|/|S|}{|B|/|S|} \equiv \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Example 1.3.3. A pair of dice is thrown. It is given that one of dice shows 4. What is the probability that the other shows 6?

$|S| = 36$, $\quad S = \left\{ \begin{matrix} (1,1), & \cdots & (1,6) \\ \vdots & & \\ (6,1), & \cdots & ,(6,6) \end{matrix} \right\}$

$B = \left\{ \begin{matrix} (4,1), (4,2), \ldots, (4,6) \\ (1,4), (2,4), (3,4) \\ (5,4), (6,4) \end{matrix} \right\}$

$A = \left\{ \begin{matrix} (6,1) & (6,2), \ldots, & (6,5), & (6,6) \\ (1,6) & \ldots & ,(5,6) & \end{matrix} \right\}$, $\quad A \cap B = \left\{ \begin{matrix} (4,6) \\ (6,4) \end{matrix} \right\}$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/36}{11/36} = 2/11$$

Example 1.3.6: Suppose we have coloured balls in three boxes in quantities given by:

	B1	B2	B3
Red	4	3	3
Green	3	3	4
Blue	5	2	3

A box is selected at random – [Uniformly choose one among: B1, B2, B3]

From that box a ball is chosen at random – [Uniformly choose one among all balls present]

Q: How likely is that a red ball is chosen?

A: $R = \{\text{red ball is chosen}\}$

$B_i = \{\text{Box } i \text{ was chosen}\}$ for $i = 1, 2, 3.$

$$\mathbb{P}(R) = ?$$

$$R = R \cap B_1 \cup R \cap B_2 \cup R \cap B_3$$ [disjoint union]

[Properties of $\mathbb{P}$] $$\mathbb{P}(R) = \mathbb{P}(R \cap B_1) + \mathbb{P}(R \cap B_2) + \mathbb{P}(R \cap B_3)$$

observe: $\mathbb{P}(B) > 0$ $$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Rewrite: $$\mathbb{P}(A \cap B) = \begin{cases} \mathbb{P}(A \mid B) \cdot \mathbb{P}(B) & \text{– if } \mathbb{P}(B) > 0 \\ 0 & \text{– otherwise} \end{cases}$$

$$\therefore \mathbb{P}(R) = \mathbb{P}(R|B_1)\,\mathbb{P}(B_1) + \mathbb{P}(R|B_2)\,\mathbb{P}(B_2) + \mathbb{P}(R|B_3)\,\mathbb{P}(B_3)$$

$$= \frac{4}{4+3+5}\ \underset{1/3}{\downarrow} + \frac{3}{3+3+2}\ \underset{1/3}{\downarrow} + \frac{3}{3+4+3}\ \underset{1/3}{\downarrow}$$

Recall :- quantity Table

	B1	B2	B3
Red	4	3	3
Green	3	3	4
Blue	5	2	3

$$\mathbb{P}(R) = \frac{4}{12}\cdot\frac{1}{3} + \frac{3}{8}\cdot\frac{1}{3} + \frac{3}{10}\cdot\frac{1}{3} = \frac{121}{360}$$

Theorem 1.3.5 :- let R be an event. let $\{B_i : 1 \le i \le n\}$ be a collection of disjoint events with $\mathbb{P}(B_i) > 0$ for all i, $1 \le i \le n$.

If $R \subseteq \bigcup_{i=1}^{n} B_i$

$$\mathbb{P}(R) = \sum_{i=1}^{n} \mathbb{P}(R|B_i)\,.\,\mathbb{P}(B_i)$$

Proof: Exercise $\square$

Question: $-\ \mathbb{P}(B) = 0$, $\quad A \cap B \subseteq B$, $\mathbb{P}(A\cap B) \le \mathbb{P}(B)$

$-\ 0 \le \mathbb{P}(A \cap B) \le \mathbb{P}(B) = 0 \longrightarrow \mathbb{P}(A \cap B) = 0$

No Problem in Convention earlier.

Example 1.3.9 [Polya's Urn]

An urn contains r - red balls & b - black balls

- A ball is drawn at random & its colour noted.
  - it is replaced with c > 0 balls of the same colour

- the procedure is repeated.

$R_1 \equiv$ red ball - was drawn in first draw

$R_2 =$ red ball - .... in 2nd draw

$B_1 \equiv$ black ball - was drawn in first draw

$B_2 \equiv$ black ball - .... 2nd draw

$$\mathbb{P}(R_1) = \frac{r}{b+r}, \quad \mathbb{P}(B_1) = \frac{b}{b+r}$$

$$\mathbb{P}(R_2) = ? = \mathbb{P}(R_2 \cap R_1 \cup R_2 \cap B_1)$$

$$= \mathbb{P}(R_2 \cap R_1) + \mathbb{P}(R_2 \cap B_1)$$

$$= \mathbb{P}(R_2 | R_1)\,\mathbb{P}(R_1) + \mathbb{P}(R_2 | B_1)\,\mathbb{P}(B_1)$$

2nd draw

$r+c$, b

$$= \frac{r+c}{r+c+b} \cdot \frac{r}{b+r} + \frac{r}{r+b+c} \cdot \frac{b}{b+r}$$

2nd draw

r, $b+c$

$$= \frac{r(r+c) + rb}{(r+c+b)(b+r)} = \frac{r(\cancel{r+b+c})}{(\cancel{r+b+c})(b+r)}$$

$$\mathbb{P}(R_2) = \frac{r}{b+r} \;!$$

It is true $\mathbb{P}(R_j) = \frac{r}{b+r} \quad \forall\, j \geq 1$

(Recall) Example 1.3.6 :- Three Boxes & Red, Green, blue balls

Recall :-

quantity Table

	B1	B2	B3
Red	4	3	3
Green	3	3	4
Blue	5	2	3

Given: Red ball is chosen.

Q :- What is Probability that Box 3 was chosen?

A : $\mathbb{P}(B_3 \mid R) = ?$

Known :- $\mathbb{P}(B_1) = \mathbb{P}(B_2) = \mathbb{P}(B_3) = 1/3$

$\mathbb{P}(R|B_1) = 4/12$, $\mathbb{P}(R|B_3) = 3/10$

$\mathbb{P}(R|B_2) = 3/8$

$\mathbb{P}(R) = \frac{121}{360}$ >0

$$\mathbb{P}(B_3|R) = \frac{\mathbb{P}(B_3 \cap R)}{\mathbb{P}(R)} = \frac{\mathbb{P}(R \cap B_3)}{\mathbb{P}(R)}$$

$$= \frac{\mathbb{P}(R|B_3) \cdot \mathbb{P}(B_3)}{\mathbb{P}(R)} = \frac{3/10 \cdot \frac{1}{3}}{121/360}$$

$$= 36/121 \quad \square$$

- observations :- From $\mathbb{P}(B_1)$, $\mathbb{P}(B_2)$, $\mathbb{P}(B_3)$
$\mathbb{P}(R|B_1)$, $\mathbb{P}(R|B_2)$, $\mathbb{P}(R|B_3)$

Computed $\longrightarrow$ $\mathbb{P}(R)$

Thus we $\longrightarrow$ calculated $\mathbb{P}(B_3|R)$

- Holds in general - Bayes Theorem

Theorem 1.3.11 [Bayes Theorem] Suppose A is an event $\{B_i : 1 \le i \le n\}$ are a collection of disjoint events whose union contains A. $\left[A \subseteq \bigcup_{i=1}^{n} B_i\right]$.

Assume $\mathbb{P}(A) > 0$, $\mathbb{P}(B_i) > 0$, for all $1 \le i \le n$.

Then for any $1 \le i \le n$,

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}{\sum_{j=1}^{n} \mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}$$

Proof:-

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(B_i \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B_i)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}{\mathbb{P}(A)}$$

$A \subseteq \bigcup_{i=1}^{n} B_i \implies A = \bigcup_{j=1}^{n} A \cap B_j$ (disjoint union)

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{j=1}^{n} A \cap B_j\right)$$

$$= \sum_{j=1}^{n} \mathbb{P}(A \cap B_j)$$

$$= \sum_{j=1}^{n} \mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)$$

$$\mathbb{P}(B_i \mid A) = \frac{\mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)}{\sum_{j=1}^{n} \mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)} \qquad 1 \le i \le n$$

$\square$

Example 1.3.12:

Shyam – tested positive for flu.

Given: 95% of people with flu test positive

but 2% of people without flu also test positive.

Known: 1% of the population has the flu.

Q: What is Probability that Shyam has the flu?

A:

A = { Shyan has flu }

B = { shyan test positive }.

$$\mathbb{P}(A) = 0.01, \quad \mathbb{P}(B|A) = 0.95$$

$$\mathbb{P}(B|A^c) = 0.02$$

$$\mathbb{P}(A|B) = ? \underset{\text{Bayes}}{=} = \frac{\mathbb{P}(B|A) \cdot P(A)}{\mathbb{P}(B|A) P(A) + P(B|A^c) P(A^c)}$$

$B_1 = B$

$B_2 = B^c$

finish next time

$$= 0.32$$

□