Recall:

Setup
$$\begin{cases} S - \text{sample space [Countable]} \\ \mathcal{F} - \text{any subset of } S - \text{event} \\ \mathbb{P} - \text{Probability} : \mathcal{F} \to [0,1] \end{cases}$$

① $P(S) = 1$; ② $\{A_j\}_{j \geq 1}$ are collection of disjoint events
$$\mathbb{P}\Big(\underbrace{\bigcup_{j=1}^{\infty} A_j}\Big) = \underbrace{\sum_{j=1}^{\infty} \mathbb{P}(A_j)}$$

Properties

- $\mathbb{P}(\phi) = 0$, $\quad A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

$n \geq 1$ - $\{E_i\}_{i=1}^{n}$ - were a collection of disjoint event
$$\mathbb{P}\Big(\bigcup_{k=1}^{n} E_k\Big) = \sum_{j=1}^{k} \mathbb{P}(E_j)$$

- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

1.2 Equally likely outcomes

Experiments :-

- Toss a fair coin < Head, tail
- Roll a fair dice < 1, 2, 3, 4, 5, 6

Observation :- S - Countable $\mathcal{F}$, $\mathbb{P} : \mathcal{F} \to [0,1]$

Suppose we know: $\mathbb{P}(\underbrace{\{\omega\}}_{\subseteq S}) = \quad \forall\ \omega \in S$

Then $A \subseteq S$,

A is countable

$$\mathbb{P}(A) = \mathbb{P}\Big(\underbrace{\bigcup_{\omega \in A} \{\omega\}}_{\text{disjoint union of countable events}}\Big) \overset{\text{Axiom ②}}{=} \sum_{\omega \in A} \mathbb{P}(\{\omega\})$$

Thus $\mathbb{P}(A)$ is determined.

If S is countable then $\{\mathbb{P}(\{\omega\}) : \omega \in S\}$ determine the function $\mathbb{P} : \mathcal{F} \to [0,1]$.

- The assignment of probabilities to each outcome is called a "distribution".

$$S = \{H, T\} \qquad \mathcal{F} = \{\phi, \{H\}, \{T\}, \{H,T\}\}$$

$$\mathbb{P}(\{H\}) = \frac{1}{2}, \qquad \mathbb{P}(\{T\}) = \frac{1}{2}$$

by above $\qquad \mathbb{P} : \mathcal{F} \to [0,1]$

$S = \{\omega_1, \dots, \omega_n\}$ – equally likely outcomes – { is standard & prevalent

Theorem 1.2.1 :- [Uniform $\{\omega_1, \omega_2, \dots \omega_n\}$] Let $S = \{\omega_1, \omega_2, \dots, \omega_n\}$ be a nonempty, finite set. If $E \subseteq S$ let

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{|E|}{n}$$

number of elements in E

Then $\mathbb{P} : \mathcal{F} \to [0,1]$ is a probability on S & assigns equal probability to each individual outcome in S

Proof:- $\mathbb{P} : \mathcal{F} \to [0,1]$ given by $\mathbb{P}(E) = \frac{|E|}{|S|}$

[well-defined]

- $E_1 = E_2 \implies \frac{|E_1|}{|S|} = \frac{|E_2|}{|S|} \implies \mathbb{P}(E_1) = \mathbb{P}(E_2)$

$E \subseteq S$, $|E| \le |S|$, $0 \le |E| \implies \mathbb{P}(E) = \frac{|E|}{|S|} \in [0,1]$

[Axiom 1 verification]

$$\mathbb{P}(S) = \frac{|S|}{|S|} = 1 \text{, trivially true.}$$

[Axiom 2 verification]

let $\{E_j\}_{j\geq 1}$ be a countable collection of disjoint events in S.

Ex:- observation: As $|S| = n$, without loss of generality, we may assume that $\{E_j\}_{j\geq 1}$

$$E_j = \phi \quad \forall j > n.$$

Then properties of probability [Theorem 1.4.4 (i)]

$$\mathbb{P}(E_j) = 0 \quad \forall j > n \quad \text{— } (*)$$

Ex:- $\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{n} E_j$; $\underbrace{\left|\bigcup_{j=1}^{n} E_j\right| = \sum_{j=1}^{n} |E_j|}_{\text{induction on } n\geq 1}$

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \mathbb{P}\left(\bigcup_{j=1}^{n} E_j\right) \overset{\text{Definition}}{=} \frac{\left|\bigcup_{j=1}^{n} E_j\right|}{|S|}$$

$$= \frac{\sum_{j=1}^{n} |E_j|}{|S|} \overset{\text{Definition}}{=} \sum_{j=1}^{n} \mathbb{P}(E_j) \overset{(*)}{=} \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

∴ Axiom (2) is verified.

[Equally likely outcomes]

$\omega \in S$, $\mathbb{P}(\{\omega\}) = \frac{|\{\omega\}|}{|S|} = \frac{1}{n}$ — $\forall \omega \in S$ $\mathbb{P}(\{\omega\}) = \frac{1}{n}$

∴ every outcome has the same probability

$\square$

Example : - A group of 12 people include Anita & Siva

- we select three people at random ← [Equally likely outcomes]

Q - How likely is it that three person group will include Anita but not Siva?

S - collection of all three person groups

- each group is as likely to be selected as any other.

$|S|$ = # of ways of choosing a 3-person group.

$$= \binom{12}{3} = \frac{12\times 10\times 11}{3\times 2\times 1} = 220$$

E - event of interest is to have Anita in group. & Siva not in the group : $\underline{\text{Anita}}$ __ __ $\equiv \binom{10}{2} = |E|$

$$|E| = \frac{10\times 9}{2\times 1} = 45$$

∴ Theorem 1.2.1 will imply

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{45}{220} \qquad \square$$

1.3 Conditional Probability

- $(S, \mathcal{F}, \mathbb{P})$ $\quad \underbrace{|S| = n}$, $\underline{\text{Uniform}\{\omega_1, \omega_2, \ldots, \omega_n\} \text{ distribution}}$

  Probability — Equally likely outcomes

Example 1.3.1 :- Toss a fair coin three times

$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$ Equally likely outcomes

A - event that there are two or more heads

$A = \{hhh, hht, thh, hth\}$

$$\mathbb{P}(A) = \frac{|A|}{|S|} = \frac{4}{8} = \frac{1}{2}.$$

B - event that there is a Head in 1^{st} toss.

$B = \{hhh, hht, hth, htt\}$

$$\mathbb{P}(B) = \frac{|B|}{|S|} = \frac{4}{8} = \frac{1}{2}.$$

Q: It is given that there is a head in 1^{st} toss then what is probability of getting two or more heads in 3-tosses?

A:- find probability of A given that B has happened.

Intuitively :- Restrict the sample space S to B

& look at occurrence of A inside B :.

Event of interest $\longrightarrow$ $A \cap B = \{hhh, hht, hth\}$

Probability of A given that B has happened

$$\equiv \frac{|A \cap B|}{|B|} = \frac{3}{4} \qquad \square$$

Occurrence of B affect the probability of A.

Definition 1.3.2 [Conditional Probability] Let S be a sample space with probability $\mathbb{P}$. Let A and B be two events with $\mathbb{P}(B) > 0$. Then the conditional probability of A given that B has happened is written $\mathbb{P}(A \mid B)$ and is defined by

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

- definition allows us to compute conditional probabilities from "unconditional probabilities".

- Remark: in Examples

$$\frac{|A \cap B|}{|B|} = \text{Computation of conditional Probability}$$

- same as definition because

$|S| < \infty$; $\quad \dfrac{|A \cap B|}{|B|} = \dfrac{|A \cap B| / |S|}{|B| / |S|} = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$