

Recall II: S - sample space [Countable]

Setup: \mathcal{F} - any subset of S - event

P - Probability : $\mathcal{F} \rightarrow [0,1]$

(1) $P(S) = 1$; (2) $\{A_j\}_{j \geq 1}$ are collection of disjoint events $P(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} P(A_j)$

Properties

- $P(\emptyset) = 0$, $A \subseteq B \Rightarrow P(A) \leq P(B)$

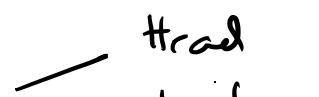
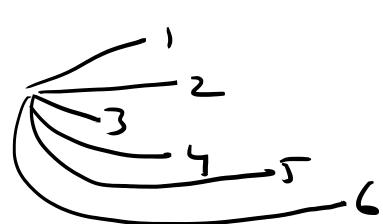
(3) - $\{E_i\}_{i=1}^n$ - over a collection of disjoint event

$$P(\bigcup_{k=1}^n E_k) = \sum_{j=1}^n P(E_j)$$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1.2 Equally likely outcomes

Experiments :-

- Toss a fair coin 
- Roll a fair dice 

Observation: S - Countable \mathcal{F} , $P: \mathcal{F} \rightarrow [0,1]$

Suppose we know $P(\{\omega\}) = \frac{1}{|S|}$ $\omega \in S$

Then $A \subseteq S$, $P(A) = P(\bigcup_{\omega \in A} \{\omega\}) \stackrel{(2)}{=} \sum_{\omega \in A} P(\{\omega\})$ Axion
disjoint union
of countable events

Thus $P(A)$ is determined.

If S is countable then $\{P(\{\omega\}) : \omega \in S\}$ determine the function $P: \mathcal{F} \rightarrow [0,1]$.

- The assignment of probabilities to each outcome is called a "distribution".

$$S = \{H, T\} \quad \mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$$

$$P(\{H\}) = \frac{1}{2}, \quad P(\{T\}) = \frac{1}{2}$$

by above

$$P: \mathcal{F} \rightarrow [0, 1]$$

$S = \{\omega_1, \dots, \omega_n\}$ - equally likely outcomes - is standard & prevalent

Theorem 1.2.1 :- [Uniform $\{w_1, w_2, \dots, w_n\}$] Let $S = \{\omega_1, \omega_2, \dots, \omega_n\}$ be a nonempty, finite set. If $E \subseteq S$ let

$$P(E) = \frac{|E|}{|S|} = \frac{|E|}{n} \quad \text{number of elements in } E$$

Then $P: \mathcal{F} \rightarrow [0, 1]$ is a probability on S & assigns equal probability to each individual outcome in S .

Proof:- $P: \mathcal{F} \rightarrow [0, 1]$ given by $P(E) = \frac{|E|}{|S|}$

[well-defined]

$$- E_1 = E_2 \Rightarrow \frac{|E_1|}{|S|} = \frac{|E_2|}{|S|} \Rightarrow P(E_1) = P(E_2)$$

$$E \subseteq S, |E| \leq |S|, 0 \leq |E| \Rightarrow P(E) = \frac{|E|}{|S|} \in [0, 1]$$

[Axiom 1 verification]

$$P(S) = \frac{|S|}{|S|} = 1, \text{ trivially true}$$

[Axiom 2 verification]

Let $\{E_j\}_{j \geq 1}$ be a countable collection of disjoint events in S .

Ex:- Observation: As $|S| = n$, without loss of generality, we may assume that $\{E_j\}_{j \geq 1}$

$$E_j = \emptyset \quad \forall j > n.$$

Then Properties of probability [Theorem 1.1.4 (i)]

$$P(E_j) = 0 \quad \forall j > n \quad - \oplus$$

$$\text{Ex:- } \sum_{j=1}^{\infty} E_j = \sum_{j=1}^n E_j ; \quad | \sum_{j=1}^{\infty} E_j | = \sum_{j=1}^n |E_j|$$

induction on $n \geq 1$

$$\begin{aligned} P\left(\sum_{j=1}^{\infty} E_j\right) &= P\left(\sum_{j=1}^n E_j\right) = \frac{|\sum_{j=1}^n E_j|}{|S|} \\ &= \frac{\sum_{j=1}^n |E_j|}{|S|} \quad \uparrow \text{Definition} \\ &= \sum_{j=1}^n P(E_j) \quad \downarrow \oplus = \sum_{j=1}^n P(E_j) \end{aligned}$$

\therefore Axiom(2) is verified.

[Equally likely outcomes]

$$\omega \in S, \quad P(\{\omega\}) = \frac{|\{\omega\}|}{|S|} = \frac{1}{n} \quad - \quad \forall \omega \in S, \quad P(\{\omega\}) = \frac{1}{n}$$

\therefore every outcome has the same probability

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Example

- A group of 12 people include Anita & Siva
- we select three people at random \leftarrow [Equally likely outcomes]
- Q - How likely is it that three person group will include Anita but not Siva?

S - collection of all three person groups
 - each group is as likely to be selected as any other

$$|S| = \# \text{ of ways of choosing a 3-person group} \\ = \binom{12}{3} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$

E - event of interest is to have Anita in group & Siva not in the group : $\underline{\text{Anita}} \subseteq \underline{\text{not Siva}} \equiv \binom{10}{2} = |E|$

$$|E| = \frac{10 \times 9}{2 \times 1} = 45$$

\therefore Theorem 1.2.1 will imply

$$P(E) = \frac{|E|}{|S|} = \frac{45}{220} \quad \square$$

1.3 Conditional Probabilities

(S, \mathcal{F}, P) $|S| = n$

Probabilities

Uniform $\{\omega, \omega_1, \dots, \omega_n\}$ distribution
 Equals likely outcomes

Example 1.3.1 :- Toss a fair coin three times
 $S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$ Equally likely outcomes

A - event that there are two or more heads

$$A = \{hhh, hht, thh, hth\}$$

$$P(A) = \frac{|A|}{|S|} = \frac{4}{8} = \frac{1}{2}.$$

B - event that there is a Head in 1st toss.

$$B = \{hhh, hht, hth, htt\}$$

$$P(B) = \frac{|B|}{|S|} = \frac{4}{8} = \frac{1}{2}.$$

Q: It is given that there is a head in 1st toss
 Then what is probability of getting two or more
 heads in 3-toss?

A:- find probability of A given that B has
 happened.

Intuitively :- Restrict the sample space S to $\underline{\mathcal{B}}$

e.g. look at $\{A\}$ inside \mathcal{B} :-
occurrence

Event of interest $\longrightarrow A \cap \mathcal{B} = \{h\bar{h}, \bar{h}h, h\bar{h}\}$

Probabilities of A given that \mathcal{B} has happened

$$\equiv \frac{|A \cap \mathcal{B}|}{|\mathcal{B}|} = \frac{3}{4} \quad \square$$

Occurrence of \mathcal{B} affect the probabilities of A .

Definition 1.3.2 [Conditional Probability] Let S be a sample space with probability P . Let A and \mathcal{B} be two events with $P(\mathcal{B}) > 0$. Then the conditional probability of A given that \mathcal{B} has happened is written $P(A | \mathcal{B})$ and is defined by

$$P(A | \mathcal{B}) = \frac{P(A \cap \mathcal{B})}{P(\mathcal{B})}$$

- definition allows us to compute conditional probabilities from "unconditional probabilities"

- Remark: in example

$\frac{|A \cap B|}{|B|} =$ Computation of Conditional Probability

- same as definition because

$$|S| < \infty; \quad \frac{|A \cap B|}{|B|} = \frac{|A \cap B| / |S|}{|B| / |S|} = \frac{P(A \cap B)}{P(B)}$$