1. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$. Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that f continuous at all $x \in \mathbb{Q}^c$ and discontinuous at all $x \in \mathbb{Q}$.

Solution(s) :-

- $f : (0,1) \to \mathbb{R}$ with the above properties
  i.e f is continuous in $\mathbb{Q}^c \cap (0,1)$
  not continuous in $\mathbb{Q}$

Ex: Extend above $f : \mathbb{R} \to \mathbb{R}$

(I) Take: $f : (0,1) \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^c \\ 1/q & \text{if } x \in p/q, \ p \in \{1, \dots, q-1\}, \ (p,q) = 1 \end{cases}$$

let $x \in \mathbb{Q}^c \cap (0,1)$. let $\varepsilon > 0$ be given

- $\exists q_0 \in \mathbb{N}$, $\frac{1}{q} < \varepsilon \quad \forall q \geq q_0$

- let $\delta > 0$

$s \in \mathbb{Q} \quad |x - s| < \delta \implies s = p/q$ and $q > q_0$

$(\because \{ p/q \mid q \leq q_0 \text{ and } p \in \{1, \dots, q-1\} \}$ is a finite set$)$

let $y \in (x - \delta, x + \delta) \cap (0,1)$

$|f(x) - f(y)| = 0 \quad y \in \mathbb{Q}^c$

$|f(x) - f(y)| < \frac{1}{q_0} \quad y \in \mathbb{Q}$

$\implies |f(x) - f(y)| < \varepsilon \quad \forall y \in (x-\delta, x+\delta)$

As $\varepsilon > 0$ was arbitrary $\implies f$ is continuous at $x \in \mathbb{Q}^c \cap (0,1)$

$x \in \mathbb{Q} \cap (0,1) \Rightarrow x = p/q \qquad (p,q)=1$
$p \in \{1, \ldots, q-1\}$

$\Rightarrow \quad f(x) = 1/q$

let $\varepsilon_0 = \frac{1}{2q}$

let $\delta > 0$ be given $\exists\, y \in \mathbb{Q}^c$
$y \in (x-\delta, x+\delta) \cap (0,1)$

$\Rightarrow \quad |f(x) - f(y)| = \frac{1}{q} \geq \varepsilon_0$

As $\delta > 0$ was arbitrary $\Rightarrow$
f is not continuous at $x \in \mathbb{Q}^c \cap (0,1)$ $\square$

(II) $(0,1)$ $\quad \mathcal{B} =$ smallest σ-algebra that contains all open intervals

let $\{q_i\}_{i \geq 1}$ be an enumeration of $\mathbb{Q} \cap (0,1)$

$\mu : \mathcal{B} \to [0,1]$ - Probability measure.

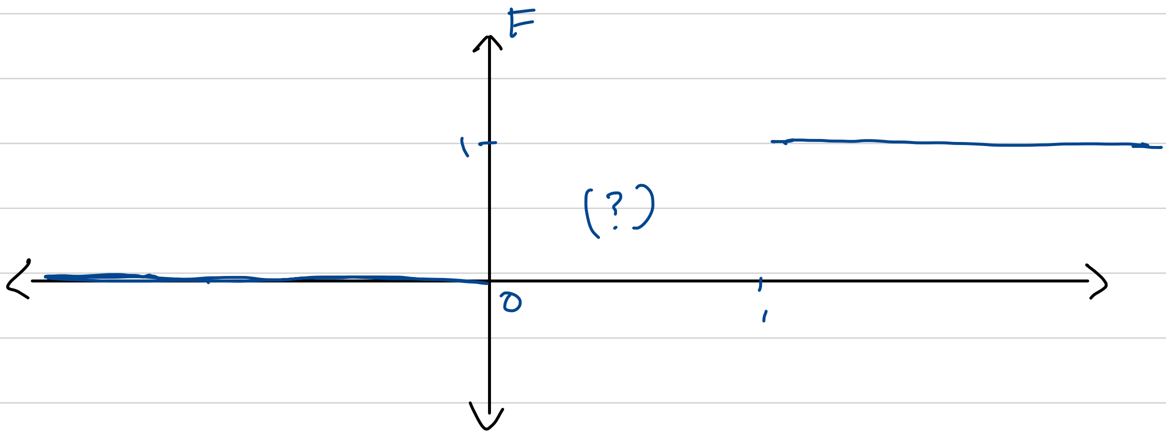
$$\mu(\{q_i\}) = \frac{1}{2^i} \qquad i \geq 1$$

$$A \in \mathcal{B} \qquad \mu(A) = \mu(A \cap \mathbb{Q})$$

$$= \sum_{q_j : q_j \in A \cap \mathbb{Q}} \mu(\{q_j\})$$

Distribution function of μ : $F : \mathbb{R} \to [0,1]$

$$F(x) = \mu((-\infty, x])$$
$$:= \mu((-\infty, x] \cap [0,1])$$



Ex:- - $F(\cdot)$ is monotonically increasing
- countably many discontinuities

Take $f : (0,1) \to \mathbb{R}$ by $f(x) = F(x)$

- $x \in \mathbb{Q}^c$ $F(\cdot)$ is continuous at $x \in \mathbb{Q}^c \cap (0,1)$
- $x \in \mathbb{Q}$ $F(\cdot)$ is discontinuous at $x \in \mathbb{Q} \cap (0,1)$

Ex:- $\mu((0,1)) = 1$ $A_i \in \mathcal{B}$

(a) $A_1 \subseteq A_2 \subseteq \cdots A_n \subseteq \cdots$ $A_i \subseteq (0,1))$

then $\mu(A_n) \longrightarrow \mu(\bigcup_{m=1}^{\infty} A_m)$.

as $n \to \infty$

Hint:- $D_n = A_n \cap (A_1^c \cap A_2^c \cdots A_n^c)$ are disjoint sequence

$\mu(\bigcup_{n=1}^{\infty} D_n) = \sum_{n=1}^{\infty} \mu(D_n)$ Relate, D_n to "A_n"

(b) $B_1 \supseteq B_2 \supseteq \dots$ $\quad B_i \subseteq (0,1)$; $B_i \in \mathcal{B}$

$(\mu((0,1)) < \infty)$ then $\mu(B_n) \longrightarrow \mu\left(\bigcap_{n=1}^{\infty} B_n\right)$

as $n \to \infty$

Hint :- Take complements and apply (a)

$x \in (0,1)$ and $n \geq 1$

$$B_n = \left(x - \tfrac{1}{n}, x\right].$$

$$B_n \supseteq B_{n+1} \quad \forall n \geq 1$$

$$\bigcap_{n=1}^{\infty} B_n = \{x\}$$

Take n - large enough $B_n \subseteq (0,1)$

(By above Result) $\mu(B_n) \longrightarrow \mu(\{x\})$ as $n \to \infty$

$$f(x) - f\left(x - \tfrac{1}{n}\right) = \mu(B_n)$$

The above implies

$$\lim_{n \to \infty} \left(f(x) - f\left(x - \tfrac{1}{n}\right)\right) = \mu(\{x\}) \quad -(*)$$

- let $x \in \mathbb{Q}^c \cap (0,1)$ $\Rightarrow \mu(\{x\}) = 0$ $\quad -(**)$

let $\varepsilon > 0$ be given

$\boxed{(*), (**)}$ $\quad \exists\ N \geq 1$ $\quad n \geq N$

$$f(x) - f\left(x - \tfrac{1}{n}\right) < \varepsilon \quad \forall n \geq N$$

$-(++)$

Now $f(x+\frac{1}{n}) - f(x) = \mu((x, x+\frac{1}{n}])$

$\tilde{B}_n = (x, x+\frac{1}{n}]$ $\quad \bigcap_{n=1}^{\infty} \tilde{B}_n = \phi$

$\Rightarrow \mu(\tilde{B}_n) \longrightarrow 0$ as $n \to \infty$.

let $\varepsilon > 0$ be given $\exists N_2 \geq 1$ such that

$|f(x+\frac{1}{n}) - f(x)| < \varepsilon \quad \forall n \geq N_2$

(++*)

(++) and (++*) $\Rightarrow$ $f(\cdot)$ is continuous at $x \in \mathbb{Q}^c \cap (0,1)$

(take $\delta = [\min(\frac{1}{N_1}, \frac{1}{N_2})] \cdot \frac{1}{2}$ and $f(\cdot)$ is monotonically increasing)

$x \in \mathbb{Q} \cap (0,1)$ $\quad x = a_{j_0} \Rightarrow \mu(\{x\}) = \frac{1}{2^{j_0}}$

$\varepsilon_0 = \frac{2}{2^{j_0}}$ $\quad \exists N_1 \geq 1$ such that

$\boxed{\text{by } (*)}$

$\forall n \geq N_1$

$|f(x) - f(x-\frac{1}{n}) - \frac{1}{2^{j_0}}| < \frac{2}{2^{j_0+1}}$

$\Rightarrow n \geq N_1$

$f(x) - f(x-\frac{1}{n}) > \frac{1}{2^{j_0+1}}$

$f(\cdot)$ is discontinuous at $x \in \mathbb{Q}^c \cap (0,1)$)

(II) – $f(\cdot)$ – is monotonically increasing
– is right continuous (HF)

did not use $x \in \mathbb{Q}^c$.

(I) is not monotonically increasing $\square$