1. Let P and Q be two probabilities on $(R, \mathcal{B})$ with $\mathcal{B}$ being the Borel-σ algebra. Let $f in \mathcal{G}$ and
$$\int f dP = \int f dQ$$
   (a) Suppose $\mathcal{G}$ was the set of all bounded real continuous function f on R. Show that P is same as Q — which means that for every Borel set B, $P(B) = Q(B)$.
   (b) Is the above true if $\mathcal{G}$ is the set of all bounded real uniformly continuous functions ?
   (c) Is the above true if $\mathcal{G}$ is the set of all bounded C^{∞} functions?
   (d) Is the above true if $\mathcal{G}$ is the set of all C^{∞} functions with compact support?

   What if P, Q were finite measures?

2. Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $s \in \mathbb{C}$ with $\Re(s) > 1$. Show that this is a differentiable function.

3. For $f \in L^1_R(\lambda)$ its Fourier transform is the function on R to C defined by
$$\widehat{f}(t) = \int e^{itx} f(x) d\lambda$$
   for $t \in R$. Show $\widehat{f}$ is continuous. If $f, g \in L^1$ then show that $\widehat{f * g} = \widehat{f}\,\widehat{g}$.

4. For a finite measure μ on R, its Fourier transform is the function on R to C defined by
$$\widehat{\mu}(t) = \int e^{itx} d\mu$$
   for $t \in R$.
   (a) Show $\widehat{\mu}$ is continuous.
   (b) For finite measures μ and ν on R, show that
$$x \mapsto \mu(B - x)$$
   is measurable.
   (c) Define $\mu * \nu(B) = \int \mu(B - x) d\nu(x)$ Show that this defines a measure on R. Show that $\widehat{\mu * \nu} = \widehat{\mu}\,\widehat{\nu}$

5. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Let $\mathcal{G}$ be the collection of all symmetric Borel sets, that is,
$$\mathcal{G} = \{B \in \mathcal{B} | x \in B \Leftrightarrow -x \in B.\}.$$
   (a) Show that $\mathcal{G}$ is a σ-field.
   (b) Show that $E(f|\mathcal{G})(x) = \frac{f(x)+f(-x)}{2}$.

6. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}^n$. Let Π be the permutation group on n points. Π acts on R^n in the obvious way:

$$\pi(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

A Borel set $B \in \mathcal{B}$ is Π-invariant if it is invariant under action of all $\pi \in \Pi$. A probability P on R^n is Π-invariant if for every Borel set $P(B) = P(\pi^{-1}(B))$ for all $B \in \mathcal{B}$ and $\pi \in \Pi$.

(a) Show that the collection $\mathcal{G}$ of Π-invariant Borel sets is a sigma-field.

(b) If P is Π-invariant, then show that $E(f \,|\, \mathcal{G})(x) = \frac{1}{n!} \sum_{\pi} f(\pi(x))$.

7. Consider probability space $(R^2, \mathcal{B}, P)$ where $dP = \varphi(x, y) d\lambda$. Define

$$\varphi_1(x) = \int \varphi(x, y) dy.$$

For each $x \in \mathbb{R}$, let

$$\psi(y \,|\, x) = \begin{cases} \frac{\varphi(x,y)}{\varphi_1(x)} & \text{if } 0 < \varphi_1(x) < \infty; \\ I_{[0,1]}(y) & \text{otherwise.} \end{cases}$$

(a) Show: for each x, as a function of y, $\psi(y \,|\, x)$ is a probability density function.

(b) For any bounded function $f(x, y)$ define $f^*(x, y) = \int f(x, y) \psi(y \,|\, x) dy$.

(c) If $\mathcal{G}$ is the σ-field generated by the map $\pi_1(x, y) = x$, show that $E(f \,|\, \mathcal{G}) = f^*$.