1. Consider a measure space $(\Omega, \mathcal{A}, \mu)$. Show that simple function based on sets of finite measure are dense in L^p for $1 \leq p < \infty$.

2. Consider $[0,1] \times [0,1]$ with Borel σ-field $\mathcal{B}$.

   (a) Show that there is a measure on $[0,1] \times [0,1]$ which is not a product measure.

   (b) Let $Q : [0,1] \times \mathcal{B} \to [0,1]$ be such that:

      i. if you fix x, as a function of A, Q is a probability on the Borel sets of $[0,1]$;

      ii. if you fix Borel set $A \subset [0,1]$ then as a function of x, Q is Borel measurable on $[0,1]$[1]

   Show that for every Borel $B \in \mathcal{B}$, the map $x \mapsto Q(x, B^x)$ is a measurable function on $[0,1]$.

   (c) Let now P be a probability on $[0,1]$. Define for $B \in \mathcal{B}$,

   $$\mu(B) = \int Q(x, B^x) dP(x).$$

   Show μ is a probability on $\mathcal{B}$.

   (d) Let f be a measurable funciton on $([0,1] \times [0,1], \mathcal{B})$. Show that

   $$x \mapsto \int f(x,y) Q(x, dy)$$

   is measurable function of x on $[0,1]$.

   (e) Show that $\int f d\mu = \int \left[\int f(x,y) Q(x,dy)\right] dP(x)$

   (f) *Extra Credit:* Every probability μ on $\mathcal{B}$ arises this way from some Q and P. that is, any concept of 'area' can be obtained by fixing notion of 'length' on x-axis and fixing notions of 'lengths' on each vertical line.

3. Linearity of R-N derivative: All measures below are sigma-finite measures on $(\Omega, \mathcal{A})$.

   (a) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then $\nu_1 + \nu_2 \ll \mu$ and $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$. Also for $c > 0$, $\frac{d(c\nu_1)}{d\mu} = c\frac{d\nu_1}{d\mu}$

   (b) If $\nu \ll \mu$ and $\mu \ll \theta$ then show $\nu \ll \theta$ and $\frac{d\nu}{d\theta} = \frac{d\nu}{d\nu}\frac{d\mu}{d\theta}$

   (c) If $\nu \ll \mu$ and $\mu \ll \nu$ then show that $\frac{d\mu}{d\nu} = 1/\frac{d\nu}{d\mu}$

   (d) If $\nu_1 \ll \mu_1$ on $(\Omega_1, \mathcal{A}_1)$ and $\nu_2 \ll \mu_2$ on $(\Omega_2, \mathcal{A}_2)$ then show that $\nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2$ on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$. Find the R-N derivative in terms of the R-N derivatives on the axes.

4. Suppose $\mathcal{A}$ is a finite σ-field and $\nu \ll \mu$ (finite measures). Give a formula for $d\nu/d\mu$. What if $\mathcal{A}$ is a countable σ-field ?

[1] Such objects as Q are called probability kernels or transition functions.

5. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $\mathcal{C}$ be a family of sets in $\mathcal{A}$.

   (a) Show that there is a set $S \in \mathcal{A}$ such that:

      i. if $C \in \mathcal{C}$ then $C \subset S$ a.e., meaning $\mu(C \setminus S) = 0$ and

      ii. if S^* has the property as in (i), then $S \subset S^*$ a.e.

      iii. such a set is unique, in the sense, if S and S' satisfy the two conditions then $S = S'$ a.e.

      iv. there is a sequence of sets $\{C_n\}$ in $\mathcal{C}$ such that $S = \cup C_n$[2]

   (b) Let Θ be a collection of finite measures such that $\nu \ll \mu$ for every $\nu \in \Theta$. Show that there is a sequence $\{\lambda_n\}$ in Θ and positive numbers $\{c_n\}$ such that $\lambda = \sum c_n \lambda_n$ is a finite measure and $\nu \ll \lambda$ for all $\nu \in \Theta$ [3]

6. Consider σ-finite measure space $(\Omega, \mathcal{B}, \mu)$. Let L^∞ be the collection of all measurable functions f such that for some number M (depending on the function), $\mu(|f| > M) = 0$.

   (a) Show L^∞ is a linear space.

   (b) For $f \in L^\infty$, define

$$\|f\|_\infty = \inf\{M > 0 : \mu(|f| > M) = 0\} = \sup\{C > 0 : \mu(|f| > C) > 0\}$$

   Show :

      i. $\|f + g\| \leq \|f\| + \|g\|$

      ii. $\|cf\| = |c|\|f\|$ and

      iii. $\|f\| = 0$ iff $f = 0$ a.e.$[\mu]$.

   (c) Show the space $(L^\infty, \| \|)$is complete.

   (d) Show that the dual space[4] of L^1 is L^∞

7. Regard log as a function on $[0, \infty]$ with $\log 0 = -\infty$, $\log \infty = \infty$. Let μ, ν, P be probabilities on $(\Omega, \mathcal{A})$ with $d\mu = f dP$; $d\nu = g dP$.

   (a) Show $\int \log(f/g) d\nu$ 'makes sense', can be $-\infty$, but not $+\infty$.

   (b) Show $\int \log(f/g) d\mu$ 'makes sense', can be $+\infty$, but not $-\infty$.

   (c) Show $\int \log(f/g) d\mu = \int \log(f/g) d\nu$ iff μ and ν are same.[5]

8. Let $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{1}{2t}(x - y)^2\}, \quad (t, x, y) \in (0, \infty) \times R \times R.$

   (a) Show that for every $y \in R$ as a function of (t, x) this function satisfies the following pde, called 'heat equation':

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}; \quad \text{for } (t, x) \in (0, \infty) \times R.$$

[2]Such a set C is called the esential-sup/esential-union of $\mathcal{C}$.

[3]All the above remains true if μ is sigma-finite. This is known as Halmos-Savage theorem.

[4]Interestingly, dual of L^∞ is NOT L^1 in general.

[5]These quantities arise in information theory and statistics.

(b) Let φ be a continuous function on R which is zero outside a bounded interval. Define

$$u(0,x) = \varphi(x) x \in R \text{ and}$$

$$u(t,x) = \int_R p(t,x,y)\varphi(y)dy, (t,x) \in (0,\infty) \times R$$

(c) Show that u is a continuous function on $[0,\infty) \times R$ and satisfies the heat equation

(d) Show that even if φ is only bounded measurable, show that u satisfies the Heat equation in $(0,\infty) \times R$.