1. Show that there are Borel sets of positive Lebesgue measure in R which do not contain any interval of positive length.

2. Consider the collection L of all real random variables on a probability space $(\Omega, \mathcal{F}, P)$. Define

$$d(X, Y) = E\left\{\frac{|X - Y|}{1 + |X - Y|}\right\}$$

   (a) For $X \in L$, define $[L] = \{[X] : Y \in L,\ P(X = Y) = 1\}$. Show that $([L], d)$ is a metric space.

   (b) Show $d(X_n, X) \to 0$ iff $P(|X_n - X| > \epsilon) \to 0$ for every $\epsilon > 0$.

3. For a finite measure space (Ω, μ) show that $L^r(\mu) \subset L^p(\mu)$ if $1 \leq p < r$. Will this be true if μ is not a finite measure ?

4. Let $X_1, \ldots, X_n$ be independent random variables on $(\Omega, \mathcal{F}, P)$.

   (a) If $X_1, \ldots, X_n$ are all discrete rvs then show that they are independent iff $P(X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n) = \prod P(X_i = a_i)$ for all numbers $a_i \in \mathbb{R}$.

   (b) Suppose $X_1, \ldots, X_n$ are absolutely continuous random variables with densities $f_1, \ldots, f_n$ then show that for all Borel $B \subset R^n$,

$$P\{\omega : (X_1(\omega), \ldots, X_n(\omega)) \in B\} = \int_B g(x_1, \ldots, x_n) \prod_{i=1}^{n} dx_i,$$

   where $g : R^n \to R$ given by $g(x_1, \ldots, x_n) = \prod f_i(x_i)$.
   Conversely, suppose that $(X_1, \ldots, X_n)$ have joint density g on R^n and it is a product: $g(x_1, \ldots, x_n) = f_1(x_1) f_2(x_2) \ldots f_n(x_n)$ for some nonnegative measurable functions of one variable. Show then that $X_1, \ldots, X_n$ are independent[1].

   (c) Suppose $X_1, X_2, \ldots X_{100}$ are independent. Let

$$Y_1 = h_1(X_1, X_2, X_{100}),\ Y_2 = h_2(X_3, X_4, \ldots, X_{77}),\ Y_3 = h_3(X_{78}, X_{89}, X_{90}, X_{91}),$$

   where h_1, h_2, h_3 are measurable on the appropriate dimensional Euclideal space. Show that Y_1, Y_2, Y_3 are independent.

[1] It is not assumed that f_i are densities.