1. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Define
$$d(A, B) = \mu(A\Delta B),$$
for $A, B \in \mathcal{A}$. For $A \in \mathcal{A}$, let $[A] := \{B \in \mathcal{A} : \mu(A\Delta B) = 0\}$ and $[\mathcal{A}] = \{[A] : A \in \mathcal{A}\}$. Show that $([\mathcal{A}], d)$ is a complete metric space.

2. Consider $L^1(R, \mathcal{B}, \mu)$ be the space of integrable functions on $\mathbb{R}$ w.r.t the measure μ. For $f, g \in L^1(R, \mathcal{B}, \mu)$, let
$$d(f, g) = \int |f - g| d\mu.$$
   (a) For $f \in L^1(R, \mathcal{B}, \mu)$, let $[f] := \{g \in L^1(R, \mathcal{B}, \mu) : f = g \text{ a.e.}\}$ and identify (with abuse of notation) $L^1(R, \mathcal{B}, \mu) := \{[f] : f \in L^1(R, \mathcal{B}, \mu)\}$. Show that $L^1(R, \mathcal{B}, \mu), d)$ is a metric space.

   (b) Suppose μ is the Lebesgue measure on $\mathbb{R}$ then show that simple functions based on intervals, i.e. finite linear combinations of indicators of (bounded) intervals, are dense in $L^1(R, \mathcal{B}, \mu), d)$. Can you show that C^∞-functions with compact support are also dense ?

3. Consider $L^1(R, \mathcal{B}, \lambda)$ with λ being Lebesgue measure and $C(\mathbb{R})$ be set of all continuous functions on $\mathbb{R}$.

   (a) Let $\varphi \in C(R)$ with compact support. For $t \in R$ define its left translate $\phi_t : \mathbb{R} \to \mathbb{R}$ by
$$\varphi_t(x) = \varphi(x + t).$$
   Show that $t \mapsto \varphi_t$ is a continuous map from R to L^1.

   (b) Let $f \in (R, \mathcal{B}, \lambda)$. For $t \in R$ define its left translate $f_t : \mathbb{R} \to \mathbb{R}$ by
$$f_t(x) = f(x + t).$$
   Show that $f_t \in L^1$ and the map $t \mapsto f_t$ is a continuous map on R to L^1.

4. Consider $(\mathbb{R}^2, \mathcal{B})$ with Lebesgue measure. Let $A = \{(x, y) : |x-y| \leq 1\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x, y) = \begin{cases} 1 & \text{if } y \geq x, (x, y) \in A \\ -1 & \text{if } y < x, (x, y) \in A \\ 0 & \text{otherwise.} \end{cases}$$
Compute $\int [\int f(x, y) dx] dy$, $\int [\int f(x, y) dy] dx$, $\int f(x, y) dx dy$.

5. Consider $\mathbb{N}^2 = \{(i, j) : i \geq 0, j \geq 0, \text{ both integers.}\}$. Let μ be the counting measure $\mathbb{N}^2$ and μ_1, μ_2 be counting measures on $\mathbb{N}$. Let $f : \mathbb{N}^2 \to \mathbb{R}$ be given by
$$f(i, j) = \begin{cases} 1 & \text{for } j = i \\ -1 & \text{for } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$
Compute $\int [\int f(i, j) d\mu_1(i)] d\mu_2(j)$, $\int [\int f(i, j) d\mu_2(j)] d\mu_1(i)$, $\int [\int f(i, j)] d\mu$

6. Consider $(\mathbb{R}^2, \mathcal{B})$ with Lebesgue measure. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{\exp\{x^2y^2\}\sin(xy)}{1+x^4y^4} & \text{for } (x,y) \in [-1,\ 1] \times [-1,\ 1] \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\int [\int f(x,y)dx]dy$, $\int [\int f(x,y)dy]dx$, $\int f(x,y)dxdy$.