1. Let Ω be the set of all strictly positive integers, with $\mathcal{A}$ being all subsets.

   (a) Let $\mu : \mathcal{A} \to [0, \infty)$ be given by $\mu(A) = |A|$ for $A \in \mathcal{A}$. Show that μ is a measure on $\mathcal{A}$.

   (b) Identify a measurable function X with the sequence $\{x_n = X(n)\}$. Show $X \in L^1(\mu)$ iff $\sum |x_n| < \infty$ and then $\int X d\mu = \sum x_n$.

   (c) More generally fix any non-negative numbers $\{w_n : n \geq 1\}$ (weights). Define : $\nu : \mathcal{A} \to [0, \infty)$ be given by $\nu(A) = \sum\{w_n : n \in A\}$. Show ν is a measure and $X \in L^1(\nu)$ iff $\sum w_n|x_n| < \infty$ and then $\int X d\nu = \sum w_n x_n$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous function which is zero outside a bounded interval. Show that the Riemann integral $R - \int f(x)dx$ equals the Lebesgue integral $L - \int f d\lambda$. [1]

3. Let $\Omega = \mathbb{R}$, and $\mathcal{A}$ be the collection of countable sets and their complements. Let $\nu : \mathcal{A} \to [0, 1]$ be a measure given by

$$\nu(A) = \begin{cases} 0 & \text{if A is countable} \\ 1 & \text{if } A^c \text{ is countable} \end{cases}$$

   Show $X : \mathbb{R} \to \mathbb{R}$ is measurable iff there is a countable set A and a number a such that $X(\omega) = a$ for all $\omega \in A^c$. Show that all measurable functions are in $L^1(\nu)$ and describe $\int X d\nu$.

4. Define for $x, y \in [0, 1)$, their sum $x \oplus y$ to be

$$\begin{array}{ll} x + y & \text{if } x + y < 1 \text{ and} \\ x + y - 1 & \text{otherwise.} \end{array}$$

   Show $x \oplus y \in [0, 1)$. This is addition modulo one. Show that for any Borel set $A \subset [0, 1)$ the set $A \oplus x = \{z \oplus x : z \in A\}$ is again Borel and further $\lambda(A) = \lambda(A \oplus x)$.

5. Consider $(\mathbb{R}, \mathcal{B})$.

   (a) We define a set A to be symmetric if: $\omega \in A \Leftrightarrow (-\omega) \in A$. Show that the collection $\mathcal{S}$ of all symmetric Borel sets is a σ-field. Show a functioin X is $\mathcal{S}$ measurable iff it is Borel and satisfies $X(\omega) = X(-\omega)$ for all points $\omega \in \mathbb{R}$.

   (b) We define a set A to be 'integer invariant' if: $x \in A \Leftrightarrow x + 1 \in A$. Show the collection of all invariant Borel sets $\mathcal{I}$ is a sigma field. Show that X is $\mathcal{I}$ measurable iff X is Borel and $X(\omega) = X(\omega + 1)$ for all $\omega \in \mathbb{R}$.

6. Let $\Omega = C[0, 1]$, the collection of all real valued continuous functions on $[0, 1]$. For $0 \leq t \leq 1$, define the evaluation map $e_t : \Omega \to R$ by $e_t(f) = f(t)$. Let $F = \{e_t : 0 \leq t \leq 1\}$. Let $\mathcal{B} = \sigma(F)$, the smallest σ-field that makes each function in F measurable.

   (a) Show that singleton subsets of Ω are in $\mathcal{B}$.

   (b) Let $A = \{\varphi \in \Omega : \max_{0 \leq t \leq 1} |\sin t - \varphi(t)| < 0.2\}$. Show that $A \in \mathcal{B}$.

[1] from now on we will use $\int f(x)dx$ to mean $\int f d\lambda$