Let Ω be a non-empty set.

1. Let $\mathcal{C}$ be any collection of subsets of Ω. Let

$$\mathcal{S}_0 = \{B : B \in \mathcal{C} \text{ or } B^c \in \mathcal{C} \text{ or } B = \Omega \text{ or } B = \emptyset\}$$

   If $\mathcal{S}$ consists of all possible finite intersections of sets in $\mathcal{S}_0$, then show that $\mathcal{S}$ is a semifield. Show finite disjoint union of sets in a semifield is a field. Conclude finite disjoint union of sets in $\mathcal{S}$ is $\mathcal{F}(\mathcal{C})$ the field[1] generated by $\mathcal{C}$.

2. Consider $(\Omega.\mathcal{A})$ a measurable space. Let $n \geq 1$ integer. Consider the following collection of subsets of Ω^n:

$$\mathcal{S} = \{A_1 \times A_2 \times \cdots \times A_n : \forall\, i \quad A_i \in \mathcal{A}\}$$

   Show that this is a semifield of subsets of Ω^n.

3. Let $\mathcal{F} = \{A : A \text{ is finite; or } A^c \text{ is finite}\}$. and $\mathcal{A} = \{A : A \text{ is countable; or } A^c \text{ is countable}\}$.
   (a) Show that $\mathcal{F}$ is a field on Ω
   (b) Show that $\mathcal{A}$ is a σ-field on Ω
   (c) Let us specialize: $\Omega = R$ and $\mathcal{A}$ as above. Define $\mu : \mathcal{A} \to [0, 1]$ by

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable and} \\ 1 & \text{otherwise.} \end{cases}$$

   Show that this is a probability on $\mathcal{A}$.
   (d) The above μ takes only two values and assigns zero to all singletons. Do you think you can get such a probability on $\mathcal{B}$ the Borel σ-field?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define for $x \in \mathbb{R}$ and $\delta > 0$;

$$O(x, \delta) = \sup\{|f(u) - f(v)| : u, v \in (x - \delta, x + \delta)\}.$$

   (a) Show that the limit: $\lim_{\delta \downarrow 0} O(x, \delta)$ exists, denote it by $O(x)$.
   (b) Show that f is continuous at x iff $O(x) = 0$. Show that for each $\epsilon > 0$, the set $\{x \in \mathbb{R} : O(x) < \epsilon\}$ is an open set[2].
   (c) Conclude that the set of continuity points is a Borel set.

5. Suppose that $(\Omega, \mathcal{A}, \mu)$ is a measure space.[3]
   (a) Let

$$\mathcal{I} = \{B \subset \Omega : (\exists\, A \in \mathcal{A})\ \mu(A) = 0, B \subset A\}.$$

   Show that

[1] Thus a field generated can be gotten in three steps. Unfortunately not so for σ-field.
[2] Here the symbol O in $O(x, \delta)$ and $O(x)$ stands for 'oscillation'.
[3] If $A \in \mathcal{A}$ and $\mu(A) = 0$ and $B \subset A$, then we feel $\mu(B) = 0$. But we may be unable to say so because $B \notin \mathcal{A}$.

i. $B_1 \in \mathcal{I}, B_2 \subset B_1$ implies $B_2 \in \mathcal{I}$ and

ii. countable union of sets in $\mathcal{I}$ is again in $\mathcal{I}$

(b) Let

$$\overline{\mathcal{A}} = \{A\Delta N : A \in \mathcal{A}; N \in \mathcal{I}\} \text{ and } \overline{\mu}(A\Delta N) = \mu(A).$$

i. $\overline{\mathcal{A}}$ is a σ-field which includes $\mathcal{A}$;

ii. $\overline{\mu}$ is a good definition(?) and is a measure on $\overline{\mathcal{A}}$;

iii. $\overline{\mu}$ extends μ;

iv. If ν is any measure on $\overline{\mathcal{A}}$ which extends μ the $\nu = \overline{\mu}$.[4]

Lebesgue measure[5] Define $F(x) = 0$ for $x \leq 0$; and $F(x) = x$ for $0 \leq x \leq 1$; and $F(x) = 1$ for $x \geq 1$.

(a) Show it is probability distribution function.

(b) If μ_0 is the corresponding probability, show that

i. $\mu(I) =$ length of I for every interval $I \subset [0,1]$,

ii. $\mu([0,1]^c) = 0$.

This is called uniform probability on $[0,1]$ or also Lebesgue measure on $[0,1]$.

(c) For any integer n positive or negative define $\mu_n(A) = \mu_0(A-n)$[6]

i. Show that $\mu_n(\cdot)$ is also a probability on the Borel σ-field $\mathcal{B}$ of the real line.

ii. Show $\mu_n((n,n+1]) = 1$ and $\mu_n((n.n+1]^c) = 0$

It is called uniform distribution on the interval $(n,n+1]$.

(d) Define now for any Borel set $A \subset R$,

$$\lambda(A) = \sum_{-\infty}^{\infty} \mu_n(A). \qquad \text{(‘two sided sum’?).}$$

i. Show that $\lambda(\cdot)$ is countably additive.

ii. Show that λ extends concept of length: i.e. for any $a < b$;

$$\lambda(a,b) = \lambda(a,b] = \lambda[a,b) = \lambda[a,b] = b-a.$$

iii. Show λ is translation invariant: $\lambda(A) = \lambda(A+x)$ for $x \in R$, $A \in \mathcal{B}$.

iv. Show $\lambda(cA) = |c|\lambda(A)$ for $c \in R$. Here $cA = \{cx : x \in A\}$

v. Show that if μ is translation invariant measure on $\mathcal{B}$ with $\mu[0,1] = 1$, then $\mu = \lambda$.

vi. Show that if μ is a translation invariant measure on $\mathcal{B}$ with $\mu[0,1] < \infty$, then μ is a multiple of λ.

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[4] The measure space $(\Omega, \overline{\mathcal{A}}, \overline{\mu})$ is called completion of the measure space $(\Omega, \mathcal{A}, \mu)$.

[6] For $n = 0$, this is the above μ_0, so no confusion.

vii. If μ is a translation invariant measure on $\mathcal{B}$ with $\mu[0,1] = \infty$, then show that $\mu(I) = \infty$ for every non-empty interval (a, b).[7]

6. Suppose that $(\Omega, \mathcal{A})$ is a measurable space and there is a set M which is important but not in $\mathcal{A}$. We shall define the σ-field generated by $\mathcal{A}$ along with M. Let

$$\mathcal{A}^* = \{(A \cap M) \cup (B \cap M^c) : \quad A, B \in \mathcal{A}\}$$

Show this is a σ-field; includes M and all sets in $\mathcal{A}$ and is the smallest such[8]

[7] This is uninteresting. Here are two $\mu(A) = |A|$; thus if A is an infinite set then $\mu(A) = \infty$. Or $\nu(A) =$ zero or infinity according as A is countable or not.

[8] If we had a probability (or any measure) on $\mathcal{A}$, can you extend it to this larger bag of sets? Is the exention Unique ?