1. Let A and B be two $k \times k$ symmetric non-negative definite matrices. Their Hadamard product is $C = ((c_{ij}))$ where $c_{ij} = a_{ij}b_{ij}$, entrywise product. Show that C is also non-negative definite.

2. Suppose we have a random variable with moments μ_k (all finite), that is, $\mu_k = E(X^k)$ for $k \geq 0$, where, of course, $\mu_0 = 1$. Show that the following matrix is non-negative definite.

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{k+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_k & \mu_{k+1} & \cdots & \mu_{2k} \end{pmatrix}.$$

3. Show that $f(x) = \frac{1}{2}e^{-|x|}$ for $-\infty < x < \infty$ is a probability density. This is called Laplace density. Let X be a random variable with this density. Show $\varphi_X(t) = \frac{1}{1+t^2}$.

4. Show $h(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ for $-\infty < x < \infty$ is a probability density. This is called Cauchy density. Let Y be a random variable with this density. Show that $\varphi_Y(t) = e^{-|t|}$. [use inverse Fourier transform!]

5. Show that the Kernel $K(s,t) = e^{-|s-t|}$ defined on $R \times R$ is non-negative definite.

6. (Measurability issues)

   (a) Consider $C[0,1]$ with sup norm topology. Show that the Borel sigmafield (that is, generated by open sets) is same as the one we described via evaluation maps.

   (b) The cointossing space 2^∞ is, under product topology, a compact metric space. Show the sigma-field we defined is its Borel sigma-field.

   (c) Is a convex set in R^2 necessarily Borel? What about convex subsets of R?

   (d) If A and B are two closed subsets of the real line, show that $A + B = \{x + y : x \in A, y \in B\}$ is Borel. This is not true if A, B were known only to be Borel, not easy to show.

   (e) If $B \subset R$ with $\lambda(B) > 0$, show that $B - B = \{x - y : x, y \in B\}$ contains a non-degenerate interval around zero.

   (f) Let $\mathcal{L}_2$ be the Lebesgue measurable sets of R^2, completion of Borel sets of R^2 w.r.t. λ^2. Let $\mathcal{L}$ be Lebesgue measurable sets of R. Assume that there are non-Lebesgue measurable sets in R. Show that $\mathcal{L}_2$ is NOT the product sigma field $\mathcal{L} \otimes \mathcal{L}$.

   (g) Can there be $f : R \to R^2$ which is one-one, onto, measurable and $f^{-1} : R^2 \to R$ is also measurable?

   (h) If f is measurable then $|f|$ is measurable. If $|f|$ is measurable, do you think f is measurable?

7. Let $X = (X_1, \ldots, X_k)$ be Gaussian vector with mean vector μ and covariance matrix Σ. Assume that Σ is non-singular. Show that X has a density and is given by

$$g(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} e^{-(x-\mu)^t \Sigma^{-1}(x-\mu)/2}; \qquad x \in R^n$$

8. Show using 'convolution formula': if $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \tau^2)$, independent then $X + Y \sim N(0, \sigma^2 + \tau^2)$.