1. Show that $L^p(R, \mathcal{B}, \lambda)$ is separable. Show that C^∞ functions with compact support are dense.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a finite or σ-finite measure space. Assume that there is a sequence of sets $\{C_n, n \geq 1\}$ such that $\mathcal{A} = \sigma(\{C_n, n \geq 1\})$ Show that $L^2(\mu)$ is separable.

3. $(X_n)_{n\geq 1}$ is a sequence of i.i.d. rvs on a space $(\Omega, \mathcal{A}, P)$ with a common law μ. For each n and ω, let $\mu_{n,\,\omega}$ be the empirical distribution of the first n variables/observations corresponding to the sample point ω. This means

$$\mu_{n,\omega}(A) = \frac{\#\{i \leq n : X_i(\omega) \in A\}}{n} \qquad A \in \mathcal{B}$$

   (a) Show that $\mu_{n,\omega} \Rightarrow \mu$ for a.e. ω.

   (b) Denote $F_n(\omega, x)$, the distribution function of $\mu_{n,\omega}$ and $F(x)$ that of μ. Then show that

      i. for a.e. ω (a.e. w.r.t. P) $F_n(\omega, x) \to F(x)$ at every continuity point x of F.

      ii. $\sup_x |F_n(\omega, x) - F(x)| \to 0$ for almost every ω.

4. Suppose $X_n \Rightarrow X$. Suppose $Y_n \to 0$ in probability. Show $X_n + Y_n \Rightarrow X$. This is known as Slutsky's theorem and very useful.