# Newton-Cotes Quadrature :-

$f : [a,b] \to \mathbb{R}$

$x_i \in [a,b] \qquad x_i \neq x_j \qquad 0 \leq i,j \leq n.$

$P_n(\cdot)$ - interpolating polynomial

$$\int_a^b f(x)\,dx = \int_a^b P_n(x) + E_n(f)$$

with $\displaystyle E_n(f) = \int_a^b \frac{f^{n+1}(\xi(x))}{(n+1)!} \prod_{j=0}^{n} (x - x_j)\,dx$

## Recap :-

$n = 1$ - Trapezoid rule , $f$ - twice differentiable

$$\int_a^b P_n(x)\,dx = (f(a) + f(b)) \frac{(b-a)}{2}$$

$$E_n(f) := -\frac{f''(c)\,(b-a)^3}{12}, \text{ for some } c \in (a,b)$$

Accuracy:- $E_n(f) = 0$ $f$ is polynomial of degree 1

---

$n = 2$ - Simpson's rule $f$ - four times differentiable

$$\int_a^b P_n(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$E_n(f) := -\frac{(b-a)^5}{2^5 \cdot 90} f''''(\xi), \text{ for some } \xi \in (a,b)$$

Accuracy:- $E_n(f) = 0$ $f$ is polynomial of degree 3.

Key :- Understood Error

- $\int_a^b \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{j=0}^{n}(x-x_j)\,dx \equiv$ Evaluated.

- Did not focus on nodes $\{x_i : 0 \le i \le n\}$.

Aim :- Increase accuracy to polynomials of degree $n+m$; for some large $m$?

i.e. precision would be $n+m$.

Consider nodes $\{x_i : i=0,\dots,n\}$

Newton's Polynomial interpolating $f$ at nodes

$$P_n(x) = \sum_{k=0}^{n} f[x_0,\dots,x_k] \prod_{i=0}^{k-1}(x-x_i)$$

Consider nodes $\{x_i : i=0,\dots n\} \cup \{t\}$

Newton's Polynomial interpolating $f$ at nodes

$$q_{n+1}(x) = \sum_{k=0}^{n} f[x_0,\dots,x_k] \prod_{i=0}^{k-1}(x-x_i) + f[x_0,\dots x_n,t] \prod_{i=0}^{n}(x-x_i)$$

$$= p_n(x) + f[x_0,\dots x_n,t] \prod_{i=0}^{n}(x-x_i)$$

$q_{n+1}(t) = f(t) \implies$

$t \in [a,b]$ , $t \notin \{x_i : i = 0,1,\dots,n\}$

$$f(t) = P_n(t) + f[x_0, \dots x_n, t] \prod_{\substack{i=0 \\ k=0}}^{n} (t - x_i)$$

$$\Rightarrow \quad E_n(f) = \int_a^b f[x_0,\dots,x_n,x] \prod_{j=0}^{n} (x - x_j)\, dx$$

is another representation.

( Here we have used the fact the integral does not change if function changes at finite number of points. )

**Question:-** Let $f : [a,b] \to \mathbb{R}$ be a polynomial of degree $n+m$ ; $\{x_i : 0 \le i \le n\} \subseteq [a,b]$

$x \in [a,b]$

<u>Exercise :-</u> $f[x_0, \dots x_n, x] = \sum_{k=0}^{m} \alpha_k x^k$ , for some $\alpha_k \in \mathbb{R}$

$$E(f) = \int_a^b f[x_0,\dots,x_n,x] \underbrace{\prod_{j=0}^{n} (x - x_j)}_{\psi(x)}\, dx$$

$$= \sum_{k=0}^{m} \alpha_k \int_a^b x^k \psi(x)\, dx$$

- Can we choose $\{x_i : 0 \le i \le n\}$ so that

$$\int_a^b x^k \psi(x)\, dx = 0 \quad \text{for } k = 0,1,\dots,m?$$

- Relation between $n$ & $m$?

Example :- $f : [-1, 1] \to \mathbb{R}$   $x_0, \dots, x_n$ - nodes

We know :-

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} P_n(x)\,dx + \int_{-1}^{1} f[x_0, \dots x_n, x]\,\psi(x)\,dx$$

$$\psi(x) = \prod_{i=0}^{n} (x - x_i)$$

Suppose $n = 2$ can we choose $x_0, x_1, x_2$ to get higher precision than 3 ?

$$q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Need :- $\int_{-1}^{1} x^k q(x)\,dx = 0$   $k = 0, 1, 2$

Choose :- $a_0 = a_2 = 0$

$\int_{-1}^{1} x^2 q(x)\,dx = \int_{-1}^{1} q(x)\,dx = 0$ for any $a_1, a_3$.

$0 = \int_{-1}^{1} x\,q(x)\,dx \Rightarrow 0 = a_1 \int_{-1}^{1} x^2\,dx + a_3 \int_{-1}^{1} x^4\,dx$

$\Rightarrow 0 = a_1 \left(\frac{2}{3}\right) + a_3 \left(\frac{2}{5}\right)$

Take : $a_1 = -3$   $a_3 = 5$

$q(x) = -3x + 5x^3$

$x_0 = -\sqrt{3/5}$

$x_1 = 0$   all zeros

$x_2 = \sqrt{3/5}$

Note $\quad q(x) = 5\left(x+\sqrt{\tfrac{3}{5}}\right)(x-0)\left(x-\sqrt{\tfrac{3}{5}}\right)$

$= 5\,\psi(x)$

$\Rightarrow \quad E(f) = 0$ if nodes are

$\left\{-\sqrt{\tfrac{3}{5}},\ 0,\ \sqrt{\tfrac{3}{5}}\right\}$ and

$f[x_0, x_1, x_2, x]$ is polynomial of degree at most $2$.

$\Rightarrow \quad E(f)$ is $0$ if nodes are

$\left\{-\sqrt{\tfrac{3}{5}},\ 0,\ \sqrt{\tfrac{3}{5}}\right\}$ if

$f$ is a polynomial of degree at most $5$!

$$\therefore \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} P_n(x)\,dx + \int_{-1}^{1} f[x_0,x_1,x_2,x]\,\psi(x)\,dx$$

$$= \sum_{i=0}^{2} f(x_i)\,w_i + E(f)$$

where $\quad w_i = \int_{-1}^{1} l_i(x)\,dx$, and

$l_i(\cdot) \equiv$ lagrange polynomial for $\{x_0, x_1, x_2\}$

$\square$

To Compute: $w_0, w_1, w_2$ - [no need to compute integrals of Lagrange polynomial]

Alternative approach :-

Observe $E(f) = 0$ for

$f(x) = 1$ , $f(x) = x$ , $f(x) = x^2$

$$\Rightarrow \left.\begin{aligned} 1 &= \int_{-1}^{1} dx = w_0 + w_1 + w_2 \\ 0 &= \int_{-1}^{1} x\,dx = -\sqrt{3/5}\, w_0 + 0.w_1 + \sqrt{\tfrac{3}{5}}\, w_2 \\ \frac{2}{3} &= \int_{-1}^{1} x^2 dx = \frac{3}{5} w_0 + 0.w_1 + \frac{3}{5} w_2 \end{aligned}\right\} (*)$$

$$\Rightarrow \quad w_0 = 5/9 \qquad w_1 = 8/9 \qquad w_2 = 5/9$$

$$\therefore \int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{3/5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right) + E(f)$$

where $E(f)$ is zero whenever $f$ is a polynomial of degree less than 5.

(*) - will be consistent due to Theorem below.

So we have seen that we can increase precision with $x_0, x_1, x_2$ to 5 by choosing nodes properly.

- Below is the main result that guarantees this.

**Theorem :-** let $q : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $n+1$ such that

$$\int_a^b x^k q(x)\,dx = 0 \quad , \quad 0 \le k \le n$$

Then :

(a) There exist $x_0, \ldots, x_n \in (a,b)$ & $q(x_i) = 0 \quad i = 0, \ldots$
$x_i \neq x_j$

(b) let $P_n(\cdot)$ - interpolating polynomial with nodes $\{x_0, x_1, \ldots, x_n\}$.

$$\int_a^b f(x)\,dx = \int_a^b P_n(x)\,dx$$ whenever $f$ is a polynomial of degree $\le 2n+1$.