Cubic Spline -

Natural boundary end point conditions :-

$f : [a,b] \to \mathbb{R}$

$\{(x_i, f(x_i)) : i = 0, .., n\}$ $\quad x_0 = a \quad x_n = b$

$S(x) = S_i(x) : x \in [x_i, x_{i+1})$

with $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$
$i = 0, \dots, n-1$

and :-

ⓐ $S_j(x_j) = f(x_j)$ & $j = 0, 1, .., n-1$

ⓑ $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ $\quad j = 0, 1, .., n-1$

ⓒ $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ $\quad j = 0, 1, .. n-2$

ⓓ $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ $\quad j = 0, 1, .. n-2$

ⓔ [Natural] $S''_0(a) = 0$, $S''_{n-1}(b) = 0$

Solved this as :- For $j = 0, 1, \dots, n-1$

- $a_j = S_j(x_j) = f(x_j)$

$b_{j+1} = b_j + h_j(c_j + c_{j+1})$

- $d_j = \dfrac{c_{j+1} - c_j}{3h_j}$

and $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the unique solution to

$$Ac = T$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & h_{n-2} & 2(h_{n-2}+h_{n-1}) & & h_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\ 0 \end{bmatrix}$$

we shall now list four other

Cubic - Splines : $\{x_i : 1 \le i \le n\}$ $f:[a,b) \to \mathbb{R}$

$$S(x) = S_i(x) \quad : x \in [x_i, x_{i+1}]$$
$$\& \; S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

- S, S', S'' continuous & $S(x_i) = f(x_i)$

Suppose : x_{i-1} x_i x_{i+1} x_{i+2}

S - cubic , interpolates f

$t \in [x_i , x_{i+1}]$

$$S(t) - f(t) = \frac{f^{4}(\xi)}{24} (t - x_{i-1})(t - x_i)(t - x_{i+1})(t - x_{i+2})$$

$h = x_{i+1} - x_i$

x_{i-1} x_i t x_{i+1} x_{i+2} $\xleftrightarrow{h}$

$\underbrace{\quad}_{\text{or } h/2}$

$$\Rightarrow \; \le \|f^4\|_\infty \frac{9}{384} h^4 \leftarrow O(h^4)$$

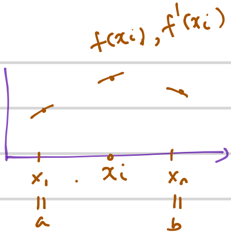
$\uparrow$ should help

- The above error analysis assumes S is a cubic polynomial in $[x_{i-1}, x_{i+2}]$ but can be shown for splines as well.

- Criteria : $f : [a,b) \to \mathbb{R}$, S - cubic spline on $[a,b]$

$$f \in C^2 \Rightarrow \int_a^b (S''(x))^2 dx \le \int_a^b (f''(x))^2 dx$$

Hermite cubic splines :-



$f : [a,b] \to \mathbb{R}$

$\text{knots} = \{x_i : i = 1, \dots, n\}$

$$S(x) = S_i(x) \qquad x \in [x_i, x_{i+1}) \qquad i = 1, \dots, n-1$$

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

& satisfy:-

$$\left.\begin{array}{ll} S_i(x_i) = f(x_i) & i = 1, \dots n-1 \\ S_i'(x_i) = f'(x_i) & \\ S_i(x_{i+1}) = f(x_{i+1}) & \\ S_i'(x_{i+1}) = f'(x_{i+1}) & \end{array}\right\} \; 4n-4$$

Solution:- For $i = 1, \dots, n-1$

$$a_i = f(x_i)$$

$$b_i = f'(x_i)$$

Exercise:-

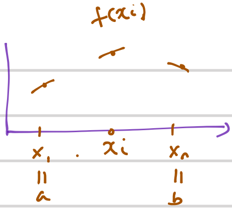
$$c_i = \frac{3f[x_i, x_{i+1}] - 2f'(x_i) - f'(x_{i+1})}{x_{i+1} - x_i}$$

$$d_i = \frac{f'(x_i) - 2f[x_i, x_{i+1}] - f'(x_{i+1})}{(x_{i+1} - x_i)^2}$$

Cubic splines :-

$f : [a,b] \to \mathbb{R}$

Knots $= \{x_i : 1 \le i \le n\}$



$S''(a) = 0 = S''(b)$

$$S(x) = S_i(x) \quad : \quad x \in [x_i, x_{i+1})$$

with $S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$ $\quad i = 1, \dots, n-1$

and :-

$$\left.\begin{array}{l} \left.\begin{array}{ll} (a) & S_j(x_j) = f(x_j) \\ (b) & S_j(x_{j+1}) = f(x_{j+1}) \end{array}\right\} \text{ for } j = 1, \dots, n-1 \\ \left.\begin{array}{ll} (c) & S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) \\ (d) & S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) \end{array}\right\} \text{ for } j = 1, \dots, n-2 \end{array}\right\} \begin{array}{c} 4n-6 \\ \text{equations} \end{array}$$

Solution :-

$$a_i = f(x_i) \qquad i = 1, \dots n-1$$

$$b_i = \dots \boxed{?}$$

$$c_i = \frac{3 f[x_i, x_{i+1}] - 2 b_i - b_i}{x_{i+1} - x_i}$$

$$d_i = \frac{b_i - 2 f[x_i, x_{i+1}] + b_{i+1}}{x_{i+1} - x_i}$$

$$\begin{bmatrix} \beta_1 & \gamma_1 & 0 & \cdot & & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & & & \vdots \\ & & \ddots & \ddots & & \\ \vdots & & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} \\ 0 & \cdots & & \cdots & \alpha_n & \beta_n \end{bmatrix} b = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{n-1} \\ \delta_n \end{bmatrix}$$

$\alpha_i = x_i - x_{i-1}$ $\quad i = 2, \dots, n-1$

$\beta_i = 2(x_i - x_{i-2})$

$\gamma_i = x_{i-1} - x_{i-2}$

$\delta_i = 3\left[f[x_i, x_{i+1}](x_{i-1} - x_{i-2}) + f[x_{i-1}, x_i](x_{i+1} - x_i)\right]$

Natural boundary :-

$S_0''(a) = 0$, $S_{n-1}''(b) = 0$

$2c_1 = 0$ $\qquad 2c_n = 0$

$b_1 + \frac{1}{2} b_2 = \frac{3}{2} f[x_1, x_2]$, $b_{n-1} + 2b_n = 3f[x_n, x_{n-1}]$

$\beta_1 = 1$, $\gamma_1 = 1/2$, $\delta_1 = \frac{3}{2} f[x_1, x_2]$

$\beta_n = 2$, $\alpha_n = 1$, $\delta_n = 3f[x_n, x_{n-1}]$

Clamped - Boundary :-

$S'(a) = f(a)$ $\qquad S'(b) = f'(b)$

$\beta_1 = 1$, $\gamma_1 = 0$, $\delta_1 = f'(a)$

$\alpha_n = 0$, $\beta_n = 1$, $\delta_n = f'(x_n)$