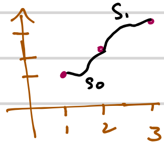
$k=3$:- Cubic Splines

- most common piecewise-polynomial approximation
- uses cubic polynomials between successive nodes

Hermite Polynomials :- H.W

Example :-

x	1	2	3
$f(x)$	2	3	5



$$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3 \quad \text{in } [1,2]$$

$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3 \quad \text{in } [2,3]$$

8 - constants
unknown

- $a_0 = S_0(1) = f(1) = 2$
  $a_1 = S_1(2) = f(2) = 3$

- $a_0 + b_0 + c_0 + d_0 = S_0(2) = f(2) = 3$
  $a_1 + b_1 + c_1 + d_1 = S_1(3) = f(3) = 5$

- S', continuous & S'' continuous

$$b_0 + 2c_0 + d_0 = S_0'(2) = S_1'(2) = b_1$$
$$2c_0 + 6d_0 = S_0''(2) = S_1''(2) = 2c_1$$

- Need two more conditions
- impose boundary conditions

Natural boundary condition :-

$$S_0''(1) = 0 \Rightarrow 2c_0 = 0$$

$$S_1''(3) = 0 \Rightarrow 2c_1 + d_1 = 0$$

Solution :-

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3 & x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3 & x \in [2,3] \end{cases}$$

$\square$

Theorem :- $f : [a,b] \to \mathbb{R}$

$$x = \left\{ (x_i, f(x_i)) \;\middle|\; \begin{matrix} i = 0, \dots n \\ x_i < x_{i+1} \end{matrix} \quad \begin{matrix} x_0 = a \\ x_n = b \end{matrix} \right\}$$

Then f has a unique cubic spline interpolant S that has $\{x_i\}_{i=0}^n$ as knots & satisfies the natural boundary conditions

1e. $S(x) = S_j(x) \quad x \in [x_j, x_{j+1}]$

$j = 0, \dots, n-1$

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

(a) $S_j(x_j) = f(x_j)$ & $j = 0, 1, \dots, n-1$

(b) $S_{j+1}(x_{j+1}) = S_j(x_{j+1}) \quad j = 0, 1, \dots, n-1$

(c) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) \quad j = 0, 1, \dots n-2$

(d) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) \quad j = 0, 1, \dots n-2$

(e) [Natural] $S''_0(a) = 0$, $S''_{n-1}(b) = 0$

4n equations

Proof:-

(a) $\Rightarrow$ $a_j = S_j(x_j) = f(x_j)$ — (1)

let $h_j = x_{j+1} - x_j \quad j = 0, \dots n-1$

(b) $\Rightarrow$

$$a_{j+1} = S_{j+1}(x_{j+1}) = S_j(x_{j+1}) = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3$$

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \quad — (2)$$

$j = 0, 1, \dots, n-1.$

(c)

$$b_{j+1} = S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) = b_j + 2c_j h_j + d_j h_j^2$$

Define: $b_n := S'_{n-1}(b) = b_{n-1} + 2c_{n-1}h_n + d_{n-1}h_n^2$

$$\Rightarrow \quad b_{j+1} = b_j + 2c_j h_j + d_j h_j^2 \quad j = 0, \dots, n-1 \qquad — (3)$$

(d) $c_{j+1} = S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) = c_j + 3d_j h_j$

$j = 0, \dots n-2$

Define: $c_n := \frac{S''_{n-1}(b)}{2} = c_{n-1} + 3d_{n-1}h_{n-1}$

$$c_{j+1} = c_j + 3d_j h_j \quad \text{for } j = 0, 1, \dots, n-1 \qquad — (4)$$

Substitute (4) into (2) [i.e, $d_j = \frac{c_{j+1} - c_j}{3h_j}$]

to get

$$a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1})$$

$j = 0, 1, \dots n-1$ — (5)

Substitute (4) into (3)

$$b_{j+1} = b_j + h_j(c_j + c_{j+1})$$

$j = 0, 1, \dots, n-1$ — (6)

Substituting (5) into (6) $\left[b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) \right]$

$j = 1, 2, \ldots, n-1$

$$h_{j-1} c_{j-1} + 2(h_{j-1} + h_j) c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}) \quad \text{— (7)}$$

Boundary Conditions.

$$\left.\begin{aligned} 2c_0 &= S_0''(a) = 0 \\ 2c_n &= S_{n-1}''(b) = 0 \end{aligned}\right\} \text{— (8)}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

7 and 8 together give

$$Ax = T \quad - \quad 9$$

as $a_{kk} > \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj}$ with $a_{ij} > 0$

$\Rightarrow$ system has a unique solution

$\therefore$ 1 — uniquely characterises $\{a_j : 0 \le j \le n-1\}$

9 — " " $\{c_j : 0 \le j \le n-1\}$

6 — " " $\{b_j : 0 \le j \le n-1\}$

4 $\{d_j : 0 \le j \le n-1\}$

$\square$