Recall:- Lagrange polynomials for interpolating points $\{(x_i, f(x_i)) : 0 \le i \le n\}$

$$l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$$p(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

- add a new point (x_{n+1}, y_{n+1}) - seems like we need to start computation from scratch.

Recursive construction:-

	x_0	x_1	x_2
x	1	-4	0
f	3	13	-23

$P_0(x) = 3$

$P_1(x) = P_0(x) + a_1(x - x_0)$

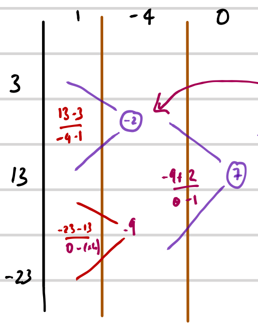
$\quad = 3 - 2(x-1)$

$\boxed{P_1(-4) = 13 \Rightarrow 3 + a_1(x-1) = 13 \Rightarrow a_1 = -2}$

$P_2(x) = P_1(x) + a_2(x - x_0)(x - x_1)$

$\quad = 3 - 2(x-1) + 7(x-1)(x+4)$

$\boxed{P_2(0) = -23 \Rightarrow -23 = 3 - 2 + a_2(-1)(4) \Rightarrow a_2 = 7}$



Newton's Polynomial :-

let (x_i, y_i) $0 \le i \le n$ be a set of points on the plane.

Consider the following :-

$$N_k(x) = \prod_{j=0}^{k-1} (x - x_j) \qquad k = 1, \dots n$$

$$N_0(x) = 1$$

$$P_n(x) = \sum_{k=0}^{n} a_k N_k(x) = \sum_{k=1}^{n} a_k \prod_{j=0}^{k-1} (x - x_j) + a_0$$

is called the Newton's polynomial of degree n.

Suppose we wish $p(x_i) = y_i$, then we get

$$y_0 = a_0$$
$$y_1 = a_0 + a_1(x_1 - x_0)$$
$$y_2 = a_0 + a_1(x_1 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$\vdots$$
$$y_n = a_0 + a_1(x_1 - x_0) + \dots + a_n \prod_{j=0}^{n-1} (x_n - x_j)$$

Need $a_0, \dots a_n$ that is a solution of

$$Na = y$$

where $a = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$ $y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & & \\ 1 & x_1 - x_0 & 0 & 0 & \cdots & & \\ 1 & x_2 - x_0 & x_2 - x_1 & 0 & \cdots & & \\ \vdots & & & & & & \\ 1 & x_n - x_0 & & \cdots & & & x_n - x_{n-1} \end{bmatrix}$$

- $$\underbrace{P_n(x)}_{} = \underbrace{P_{n-1}(x)}_{\text{determinable from } x_0, \ldots, x_{n-1}} + \underbrace{a_n \prod_{j=0}^{n-1} (x - x_j)}_{\text{from } x_n}$$

<u>To Compute:</u> a
- One can try to invert N
- Upper Dar matrix, so solution is easier.

Recall <u>Divided Differences</u> :-

$f : [a,b] \to \mathbb{R}$

let $x_0, x_1, \ldots, x_n$ be distinct real numbers in $[a,b]$

Then $f[x_0] = x_0$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

For $k \geq 2$

$$f[x_0, x_1, x_2, \ldots x_{k-1}, x_k]$$

$$= \frac{f[x_1, x_2, \ldots x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

Interpolation :-

Let $f : [a,b] \to \mathbb{R}$

let $\{(x_i, f(x_i)) : 0 \leq i \leq n\}$ be a set of points and we wish to find polynomial $P(\cdot)$ of degree at most n such that

$$P(x_i) = f(x_i) \qquad 0 \leq i \leq n$$

Example :

$n=3$ - take Newton's Polynomial

$$P_0(x) = a_0$$

$$P_1(x) = a_0 + a_1(x - x_0)$$

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

Clearly $a_0 = f(x_0) \quad - (1)$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1] \quad - (2)$$

$$a_2 = \ldots \qquad = f[x_0, x_1, x_2]$$

$\therefore$ The Newton's Polynomial of degree atmost 3 interpolating $\{ (x_i, f(x_i)) : 0 \le i \le 3 \}$

is

$$P_3(x) = \sum_{k=0}^{3} f[x_0, x_1, \dots x_k] \prod_{j=0}^{k-1} (x - x_j)$$

<u>Exercise</u> :- one can establish that

The Newton's Polynomial of degree atmost n interpolating $\{ (x_i, f(x_i)) : 0 \le i \le n \}$

is

$$P_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \dots x_k] \prod_{j=0}^{k-1} (x - x_j)$$

o $\sigma : \{0, 1, \dots, n\} \to \{0, 1, \dots, n\}$ be a Permutation.

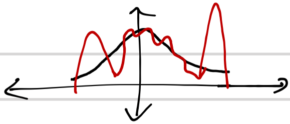
$$X = \{ (x_i, y_i) : 0 \le i \le n \} = \{ (x_{\sigma(i)}, y_{\sigma(i)}) : 0 \le i \le n \}$$

So the interpolating polynomials are same

$$\Rightarrow f[x_0, \dots, x_n] = f[x_{\sigma(0)}, \dots, x_{\sigma(n)}]$$

Polynomial wiggle :-

- $f : [-5, 5] \to \mathbb{R}$
  $f(t) = \frac{1}{1+t^2}$ in



HW7

- Runge - phenomena

- higher degree polynomials oscillate more.

- $f : [-5,5] \to \mathbb{R}$, one can try for $f(x) = |x|$

- **Recall result shown earlier**

Theorem 1 :- $f : [a,b] \to \mathbb{R}$ f be $(n+1)$ times differentiable.
Let $\{x_i : 0 \le i \le n\}$ be $n+1$ - distinct points in $[a,b]$
Let $p(\cdot)$ be the Polynomial interpolating $\{(x_i, f(x_i)) : 0 \le i \le n\}$.
Then $\forall x \in [a,b]$ $\xi(x)$ between $x_0, .. x_n$ such that

$$f(x) = p(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \cdot \underbrace{\prod_{i=0}^{n} (x - x_i)}$$

- already seen how one can choose x_i - to control Errors.

- Though exact at x_i, the approximation may fail between x_i's

Also in

- Integration - $\left\{\begin{array}{ll} \text{Simpson} & - \text{degree } 2 \\ \text{Midpoint} & - \text{degree } 0 \\ \text{Trapezoid} & - \text{degree } 1 \end{array}\right.$ Polynomial

  - introduced Composite method
    - did not do n^{th} degree interpolation

Piecewise Interpolation :-

A $S : [a,b] \to \mathbb{R}$ is said to be spline of degree k if

(a) $S, S', \ldots S^{(k-1)}$ are continuous on $[a,b]$

(b) $a = t_0 \leq t_1 \leq \ldots \leq t_n = b$ $\{t_i\}_{i=0}^{n}$ called Knots

such that S - polynomial of degree at most k in each $[t_i, t_{i+1}]$ $0 \leq i \leq n-1$

Given

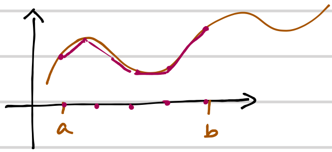
$f : [a,b] \to \mathbb{R}$

We say a Spline of degree k with Knots $\{x_i : 0 \leq i \leq n\}$ interpolates f if

$$S(x_i) = f(x_i) \qquad 0 \leq i \leq n.$$

- Uniqueness :- for $k \geq 2$ we need to impose more conditions.

$K=1$:- [piecewise linear interpolation]



$f : [a,b] \to \mathbb{R}$

S - Spline of degree 1 is the piecewise linear interpolation of $\{(x_i, f(x_i)) \mid 0 \le i \le n\}$

Notes:-

- linear approximation
  - may not be differentiable at end points.
  - Have used it in Composite Trapezoid rule.

$K=2$:- [piecewise Quadratic interpolation]

One can construct in each knots-intervals
- $[x_i, x_{i+1}]$ - Quadratic polynomial
  - Quadratic agrees at end points
  - need one more condition?

(There is flexibility - even under the requirement of smoothness)

- Alas there is no good acceptable way