

Recall

$$f: [a, b] \rightarrow \mathbb{R}$$

$$X = \{ (x_i, f(x_i)) \mid 0 \leq i \leq n \} \subseteq [a, b]$$

[Lagrange]

$$P_n(x) = \sum_{i=0}^n f(x_i) l_i(x)$$

with $l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(x - x_j)}{(x_i - x_j)}$

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b P_n(x) dx + E_n^{a,b,f} \\ &= \sum_{i=0}^n f(x_i) \int_a^b l_i(x) dx + E_n^{a,b,f} \end{aligned}$$

$$n=0$$

[Endpoint]
Rule

$$x_0 = \frac{a+b}{2}$$

polynomial P_0

$$\int_a^b f(x) dx = (b-a) f\left(\frac{a+b}{2}\right) + E_{a,b,f,0}$$

$$n=1$$

[Trapezoid]
Rule

$$x_0 = a, x_1 = b$$

Polynomial P_1

$$\int_a^b f(x) dx = \frac{(f(a) + f(b))(b-a)}{2}$$

$$- \frac{f''(c)(b-a)^3}{12}$$

→ (*)

$$n=2$$

$$x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$$

Polynomial P_2

$$\int_a^b f(x) dx = \frac{b-a}{3} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] + E_{a,b,f,2}$$

Error bounds for Midpoint & Simpson

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^n (x-x_i) \quad [\text{Error for interpolation}]$$

Approach 1

$$\int_a^b f(x) dx = \int_a^b P_n(x) dx + \underbrace{\int_a^b \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^n (x-x_i) dx}_{\text{Error estimate}}$$

Trapezoid Rule is the MVT for integrals gave the answer $\otimes$

Error formula for Midpoint Rule

We want to understand $f: [a, b] \rightarrow \mathbb{R}$, f'' is continuous

$$\text{Error}_{\text{midpoint}} = \int_a^b f(x) dx - (b-a) f\left(\frac{a+b}{2}\right)$$

$f: [0, \frac{b-a}{2}] \rightarrow \mathbb{R}$ by

$$F(t) = \int_{\frac{a+b}{2}-t}^{\frac{a+b}{2}+t} f(s) ds - 2t f\left(\frac{a+b}{2}\right)$$

$$\frac{a+b}{2} - t$$

$$\left[\text{Error}_{\text{midpoint}} = F\left(\frac{b-a}{2}\right) \right]$$

Note

$$E(0) = 0 - (1)$$

$$E'(t) = f\left(\frac{a+b}{2} + t\right) + f\left(\frac{a+b}{2} - t\right) - 2f\left(\frac{a+b}{2}\right)$$

$$\Rightarrow E'(0) = 0 - (2)$$

$$E''(t) = f'\left(\frac{a+b}{2} + t\right) - f'\left(\frac{a+b}{2} - t\right)$$

$$\Rightarrow E''(0) = 0 - (3)$$

$$E'''(t) = f''\left(\frac{a+b}{2} + t\right) + f''\left(\frac{a+b}{2} - t\right)$$

Taylor's Theorem with Integral Remainder

and ①, ②, ③

$$E(t) = \int_0^t E''(s) \frac{(t-s)^2}{2} ds$$

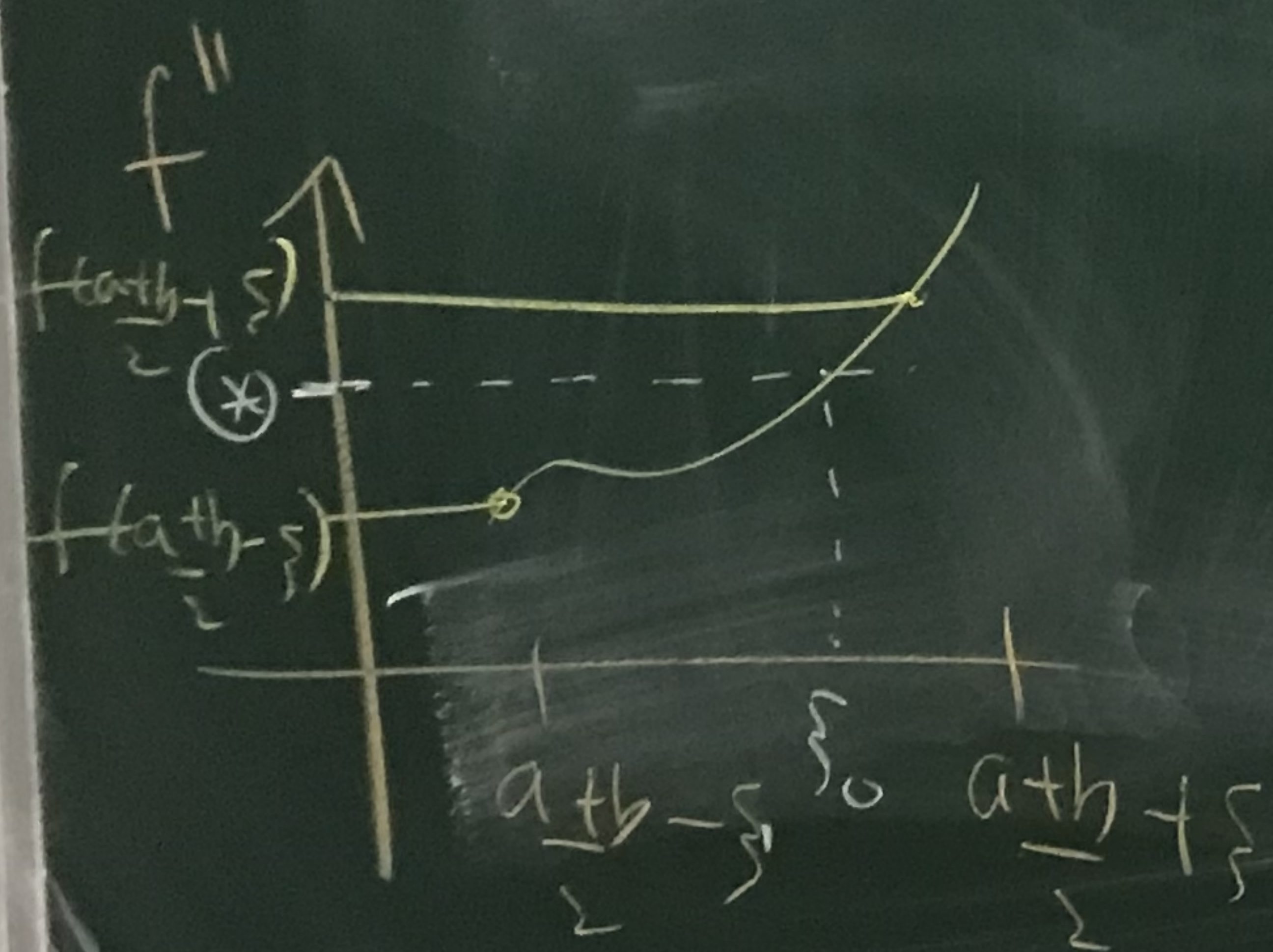
$$\left[\exists \xi \in \left[0, \frac{b-a}{2}\right] \right]$$

M.V.T
for integrals

$$\leftarrow E'''(\xi) \int_0^t \frac{(t-s)^2}{2} ds$$

$$= \left[f''\left(\frac{a+b}{2} + \xi\right) + f''\left(\frac{a+b}{2} - \xi\right) \right] \frac{t^3}{6}$$

$$= \left[\frac{1}{2} f''\left(\frac{a+b}{2} + \xi\right) + \frac{1}{2} f''\left(\frac{a+b}{2} - \xi\right) \right] \frac{t^3}{3}$$



$$\begin{aligned} & \exists \xi_0 \in [0, t] \\ & E(t) = f''(\xi_0) \frac{t^3}{3} \\ & \Downarrow \\ & E_{\text{midpoint}} = E\left(\frac{b-a}{2}\right) \\ & \parallel \\ & \frac{f''(\xi_0) (b-a)^3}{24} \end{aligned}$$

Error formula for Simpson's Rule

We want to understand $f: [a, b] \rightarrow \mathbb{R}$

f - 4 times differentiable

$$E_{\text{Simpson}} = \int_a^b f(x) dx - \frac{(b-a)}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Define $E: [0, \frac{b-a}{2}] \rightarrow \mathbb{R}$

$$E(t) = \int_{a-t}^{a+t} f(x) dx - \frac{t}{3} \left[f\left(\frac{a+b}{2}-t\right) + 4f\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}+t\right) \right]$$

Sketch:

$$E(0) = E'(0) = E''(0) = 0$$

Step 1:

$$E'''(t) = \frac{t}{3} \left[f''\left(\frac{a+b}{2} - t\right) - f''\left(\frac{a+b}{2} + t\right) \right]$$

Step 2: $\left[\begin{array}{l} \text{Taylor's Theorem} \\ \text{MVT for Integrals} \end{array} \right]$

to get: $E(t) = \int_{\xi \in [0,t]} f^{(4)}(\xi) \frac{t^5}{90}$

$$E_{\text{error Simpson}} = \frac{f^{(4)}(\xi) (b-a)^5}{2^5 90}$$

$f: [a, b] \rightarrow \mathbb{R}$
 is 4-times differentiable

$$\int_a^b f(x) dx = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] + E_{\text{error Simpson}}$$

Consequence of Error formula

Simpson's Rule is Exact value
 of Integral

for all polynomials of
 degree $n=0, 1, 2, 3$

Q: How to derive good approximation for large intervals?

Composite Rules

$$f: [a, b] \rightarrow \mathbb{R} \quad n \geq 1$$

Divide $[a, b]$ into n Equal parts

$$x_i = a + \left(\frac{b-a}{n} \right) i \quad (i = 0, \dots, n)$$

Midpoint & Simpson - Composite Rule

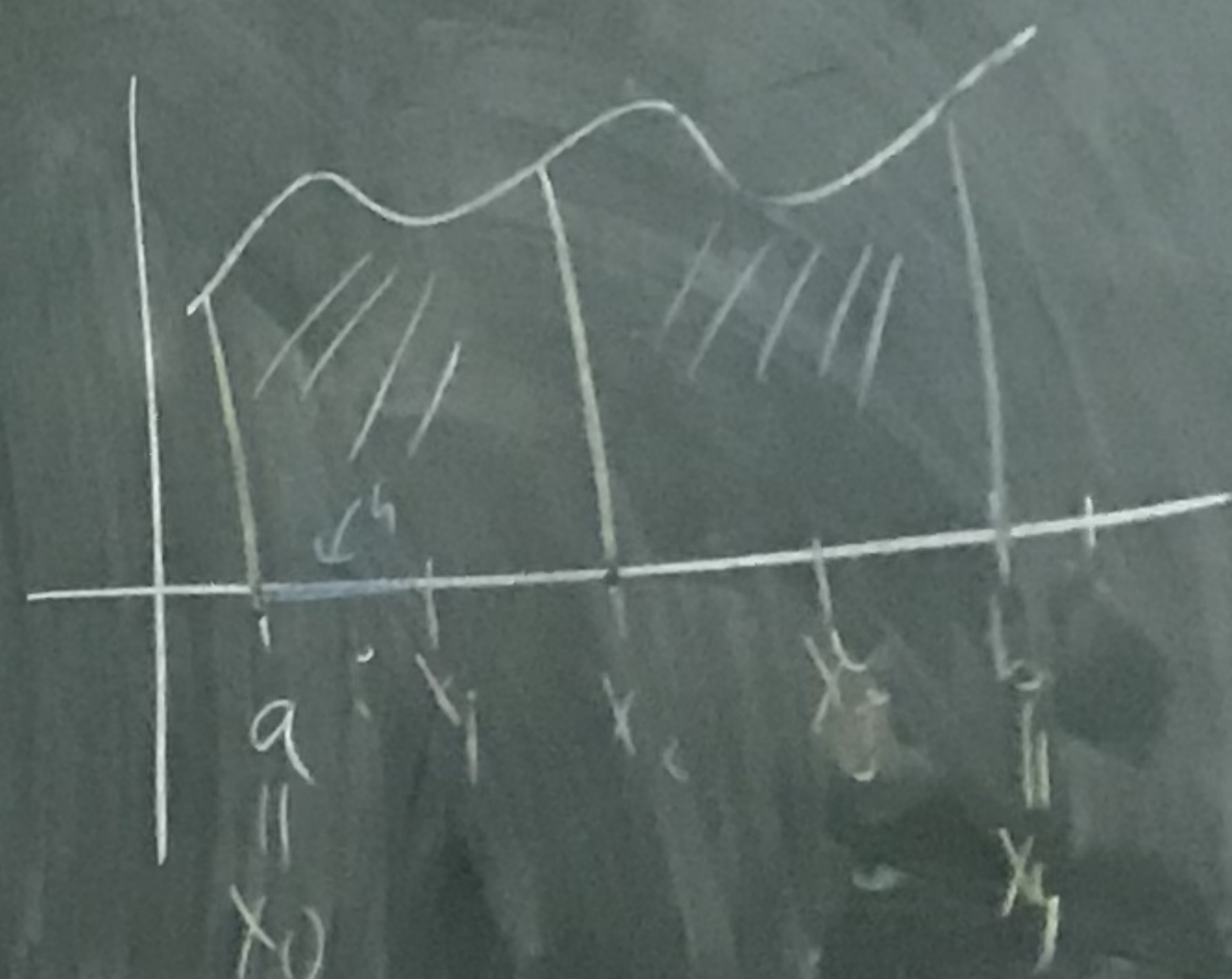
— n — even number

$$h = \frac{b-a}{n} = x_k - x_{k-1}$$

— Apply the rules to each $[x_{2j-2}, x_{2j}]$ $j=1, \dots, \frac{n}{2}$

& sum the approximations to achieve to $\int_a^b f(x) dx$
in each approximation

$$I = \int_a^b f(x) dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x) dx$$



Midpoint Rule

$$I = \sum_{j=1}^{n/2} \left[f\left(\frac{x_{2j} + x_{2j-2}}{2}\right) (2h) + \frac{(2h)^3}{12} f''(\xi_j) \right]$$

$$= 2h \sum_{j=1}^{n/2} f\left(\frac{x_{2j} + x_{2j-2}}{2}\right) + \frac{8h^3}{12} \sum_{j=1}^{n/2} f''(\xi_j)$$

$$= \frac{2(b-a)}{n} \sum_{j=1}^{n/2} f\left(\frac{x_{2j} + x_{2j-2}}{2}\right) + \frac{8(b-a)^3}{12 \cdot h^3} \frac{1}{2} \sum_{j=1}^{n/2} f''(\xi_j)$$

$$\exists \xi_0 \in [a, b]$$

$$I = 2\left(\frac{b-a}{n}\right) \sum_{j=1}^{n/2} f\left(\frac{x_j + x_{j-2}}{2}\right) + \frac{(b-a)^3}{3n^2} f''(\xi_0)$$

Composite Simpson's Rule

$$I = \frac{b-a}{3n} \left[f(a) + 2 \sum_{j=2}^{n/2} f(x_{2j-2}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right]$$

$$\frac{(b-a)^5}{n^4 180} f^{(4)}(\xi_0)$$

Composite Trapezoid Rule

$$I = \frac{b-a}{2n} \left[f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right] + \frac{(b-a)^3}{6n^2} f''(\xi_0)$$

Midterm - Monday - 24/2/2020

- Two hour test

- Proofs of Error bounds
 | Convergence results

- Will not be
 part of
 syllabus

- Tuesday - 18-2-2020 $\leq$ Material covered.

Composite Simpson rule :-

$$f: [a, b] \rightarrow \mathbb{R} \quad b-a \text{ not small}$$

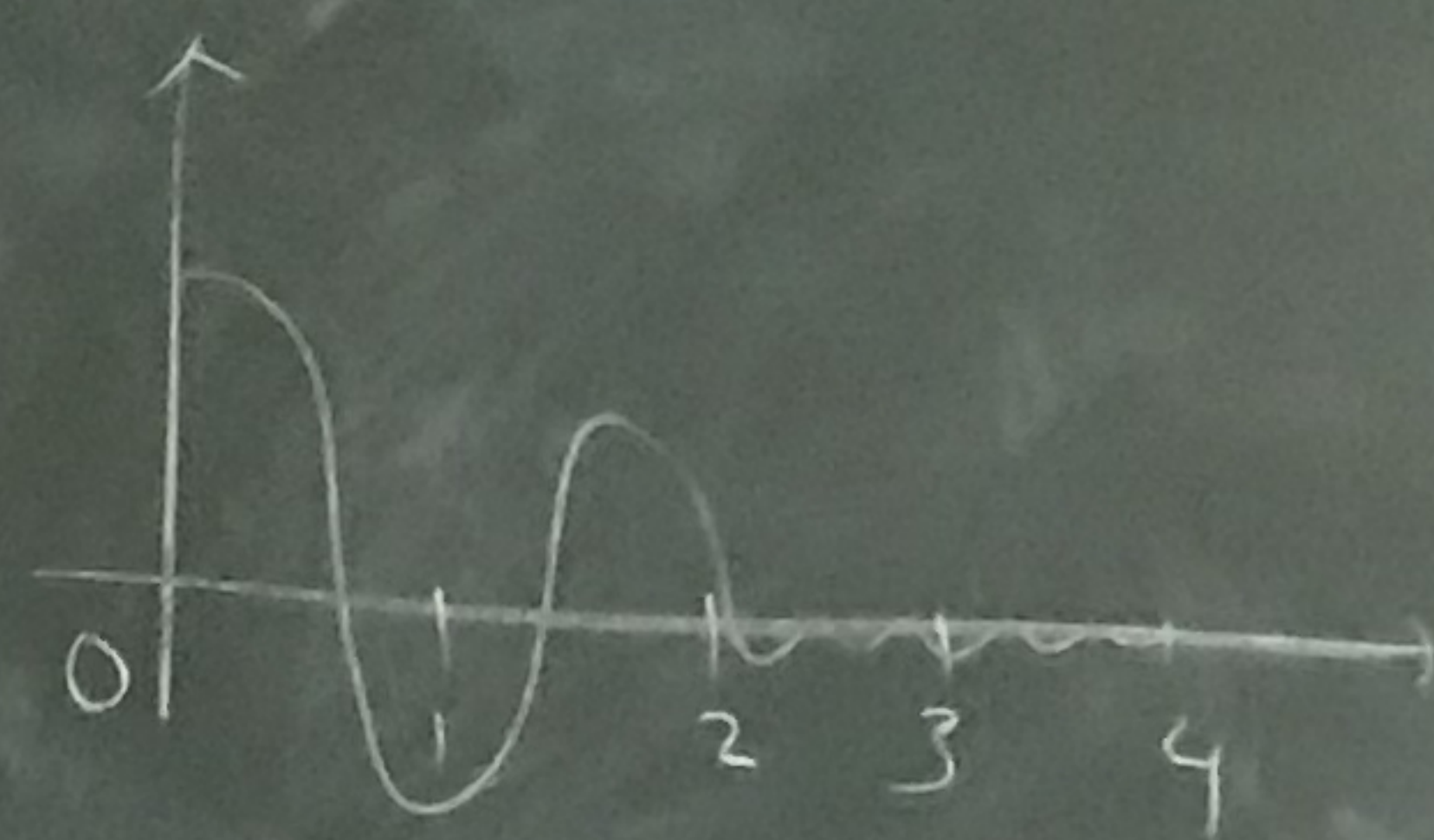
$$x_i = a + \left(\frac{b-a}{n}\right)i \quad i=0, \dots, n, \quad n-\text{even}$$

$[x_{2j-2}, x_{2j-1}]$ - apply Simpson's rule for $j=1, \dots, \frac{n}{2}$
& add approximations

$$\int_a^b f(x) dx = \frac{b-a}{3n} \left[f(a) + 2 \sum_{j=2}^{n/2} f(x_{2j-2}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right] + \frac{(b-a)^5}{180 \cdot n^4} f^{(4)}(\xi)$$

Example (Metrica media Adapted)

$$f(x) = e^{-3x} \cos(x) \quad x \in [0, 4]$$



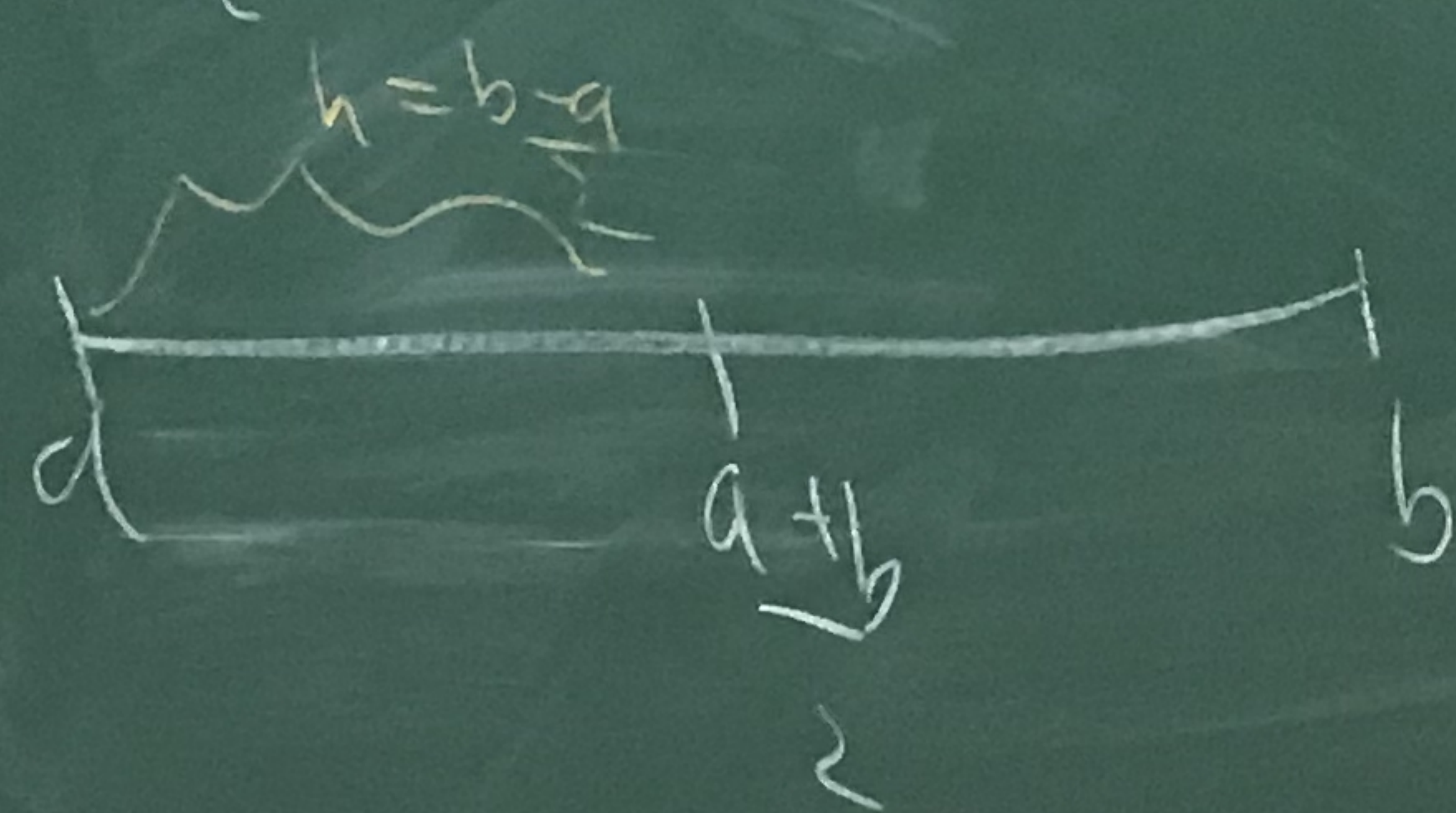
Adaptive Simpson rule

- [Need] Composite Simpson rule uses equally spaced points. This can be inappropriate depending on region where the function has large variations and small variations.

— How does one address the above?

$[a, b]$

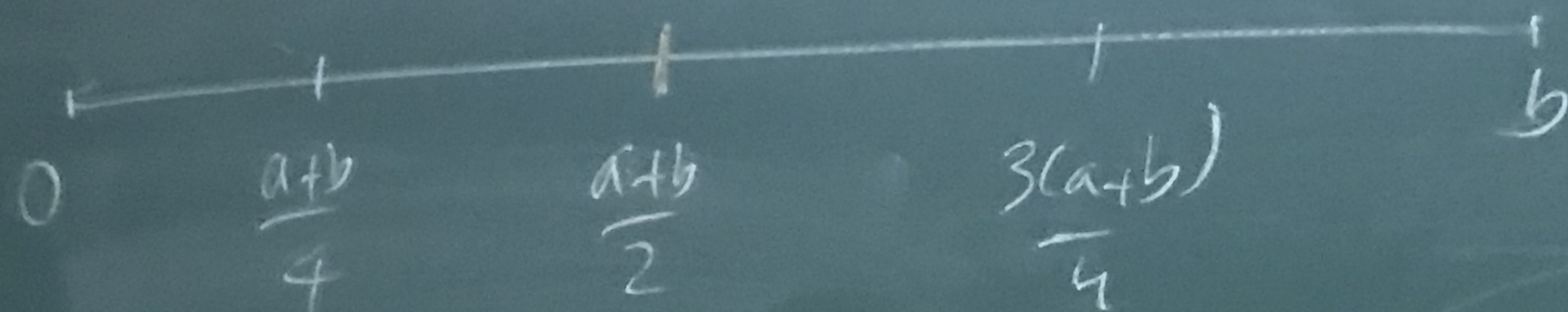
$n=2$



$$\int_{(a,b)} = \frac{h}{3} [f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

$$I - S^{(a,b)} = \left(-\frac{h^5}{90} + \frac{4}{3} \right) = E^{(1)}$$

$$\underline{n=4}$$



$$I = \int_{(a, \frac{a+b}{2})} + \int_{(\frac{a+b}{2}, b)} - \left(\frac{1}{16} \frac{h^5}{90} f^{(4)}(\xi_2) \right) = E^{(2)}$$

$$h = \frac{b-a}{2}$$

$$\left(\int_{(a,b)} - \left(\int_{(a, \frac{a+b}{2})} + \int_{(\frac{a+b}{2}, b)} \right) \right) = \frac{1}{16} \frac{h^5}{90} f^{(4)}(\xi_2) - \frac{1}{16} \frac{h^5}{90} f^{(4)}(\xi_1)$$

$$\left[\text{Suppose } \xi_1 = \xi_2 \text{ or } f^{(4)}(\cdot) = \text{constant} \right] = \frac{1}{5} E^{(2)}$$

We know

$$I = \int_{a,b}^{(1)} + E^{(1)}$$

$$I = \int_{a, \frac{a+b}{2}}^{(2)} + \int_{\frac{a+b}{2}, b}^{(2)} + E^{(2)}$$

Where $E^{(1)} = -\frac{h^5}{90} f^{(4)}(\xi)$
 $\xi \in (a,b)$

One can use the following tolerance test

$$I + \frac{1}{15} \left| \int_{a,b}^{(2)} - \left[\int_{a, \frac{a+b}{2}}^{(2)} + \int_{\frac{a+b}{2}, b}^{(2)} \right] \right| < \text{tolerance value}$$

assuming $E^{(2)} = \frac{1}{16} E^{(1)}$

(*) if $f^{(4)}(\cdot)$ is bounded
 $|E^{(2)}| < \frac{1}{16} |E^{(1)}|$

Then accept $\int_{a,b}^{(2)}$ as an approximation for I .
 otherwise refine $(a, \frac{a+b}{2})$, $(\frac{a+b}{2}, b)$ further & proceed as above

Algorithm / Pseudocode for Adapt Simpson

adapt simpson = function($f, a, b, \epsilon, \text{level}, n$)

level = level + 1

$$h = \frac{b-a}{2} ; c = \frac{a+b}{2}$$

$$S_1 = \frac{h}{3} [f(a) + 4f(c) + f(b)]$$

$$d = \frac{a+c}{2} \quad e = \frac{c+b}{2}$$

$$S_2 = \frac{h}{6} [f(a) + 4f(d) + 2f(c) + 4f(e) + f(b)]$$

if { level $\geq$ n } { I = S2 end }

if $\left(\frac{|S1 - S2|}{15} < \varepsilon \right)$ { I = $S2 + \frac{S2 - S1}{15}$; end }

else { c = $a + \frac{b-a}{2}$

I = adaptsimpson(f, a, c, $\frac{\varepsilon}{2}$, level, n)

+ adaptsimpson(f, c, b, $\frac{\varepsilon}{2}$, level, n) }

end