Recall :

- Interpolating Polynomials

$f : [a,b) \to \mathbb{R}$ $\quad X = \{ (x_i, f(x_i)) \mid 0 \le i \le n, \ x_i \in [a,b], \ x_i \ne x_j \}$

$$l_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{(x - x_j)}{x_i - x_j}$$

$$p(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$

$\Big\}$ Lagrange polynomials

• Numerical (Quadrature) Integration

$\int_a^b f(x)\,dx$ - approximated by - $\sum_{i=1}^{n} a_i f(x_i)$

$$\int_a^b p(x)\,dx = \sum_{i=0}^{n} \left(\int_a^b l_i(x)\,dx \right) f(x_i)$$

$$\text{Error} = I - \int_a^b p(x)\,dx$$

$$f(x) - p(x) = \frac{f^{n+1}(\xi(x)) \prod_{i=0}^{n} (x - x_i)}{(n+1)!}$$

Error Analysis for Midpoint-Rule

$$\text{Error}_{\text{midpoint}} = \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)$$

$$E(t) = \int_{\alpha-t}^{\alpha+t} f(s)\,ds - 2t\,f(\alpha)$$

$$E'(t) = -2f(\alpha) + f(\alpha+t) + f(\alpha-t)$$

$$E''(t) = f'(\alpha+t) - f'(\alpha-t)$$

$$E'''(t) = f''(\alpha+t) + f''(\alpha-t)$$

$$E(0) = E'(0) = E''(0) = 0$$

$$E(t) = \int_0^t E'''(s)\,\frac{(t-s)^2}{2}\,ds$$

[Taylor's Thm Integral form] Remainder

$$E(t) = \frac{E'''(\xi)}{2}\int_0^t (s-t)^2\,ds$$

$$= \frac{t^3}{6}\left[f''(\alpha+\xi) + f''(\alpha-\xi)\right]$$

$$= \frac{t^3}{3}\cdot f''(\xi_0)$$

$$\text{Error}_{\text{midpoint}} = E\left(\frac{b-a}{2}\right) = \frac{(b-a)^3}{24} f''(\xi_0)$$

Error Analysis for Simpson's rule

$f : [a,b] \to \mathbb{R}$ $\qquad X = \{ a, \frac{a+b}{2}, b \}$

$$p(x) = \sum_{i=0}^{3} l_i(x)$$

$$\text{Error}_{\text{Simpson}} = \int_a^b f'''(\xi(x)) \frac{(x-a)\left(x - \frac{(a+b)}{2}\right)(x-b)}{6} dx$$

Approach 1 :-

$f'''(\cdot)$ is bounded $\quad , h = \frac{b-a}{2}$

$\cdot \text{Error}_{\text{Simpson}} = O(h^4) \quad$ as $h \to 0$

Approach 2 :-

$\frac{b-a}{2} = h$ $\qquad x_0 = a$

$x_1 = a + h = \frac{a+b}{2}$

$x_2 = a + 2h = b$

$$E(t) = \int_{\alpha - t}^{\alpha + t} f(s)\,ds - \frac{t}{3}\left[f(\alpha - t) + 4f(\alpha) + f(\alpha + t) \right]$$

$0 \le t \le \frac{b-a}{2} \qquad \alpha = \frac{a+b}{2}$

$$\text{Error}_{\text{Simpson}} = E\left(\frac{b-a}{2}\right)$$

Verify: $E(0) = E'(0) = E''(0) = 0$

$$E'''(t) = \frac{t}{3}\left[f'''(\alpha - t) - f'''(\alpha + t) \right]$$

[Taylor's Theorem] - integral form remainder

$$E(s) = 0 + 0 + 0 + \int_0^s E'''(t)\,\frac{(t-h)^2}{2}\,dt$$

$$= \frac{1}{6}\int_0^s t\,(t-h)^2\left[f'''(\alpha-t) - f'''(\alpha+t)\right]dt$$

$$F : \left[0, \frac{b-a}{2}\right] \to \mathbb{R}$$

$$F(t) = \begin{cases} \dfrac{f'''(\alpha-t) - f'''(\alpha+t)}{t} & t = 0 \\ -2 f''''(\alpha) & t \neq 0 \end{cases}$$

Verify: F is continuous

$$E(s) = \frac{1}{6}\int_0^s t^2 (t-h)^2\, F(t)\,dt$$

[Mean-value Theorem for integrals]

$$= \frac{1}{6} F(\eta) \int_0^s t^2 (t-h)^2\, dt$$

for some $\eta \in [0, s]$

$$= \frac{f'''(\alpha-\eta) - f'''(\alpha+\eta)}{6\,\eta}\;\frac{s^5}{30}$$

$$= -\frac{s^5}{90} f''''(\xi) \qquad \xi \in (\alpha-s, \alpha+s)$$

Theorem:- $f : [a,b] \to \mathbb{R}$ be four times differentiable

let
$$\text{Error}_{\text{Simpson}} = \int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Then $\exists\ \xi \in (a,b)$ such that

$$\text{Error}_{\text{Simpson}} = -\frac{(b-a)^5}{2^5 \cdot 90} f''''(\xi) = O(h^5)$$

Remark:- Consequences of Error analysis

① Midpoint Rule & Trapezoid Rule :- This gives exact answer for all linear functions.

② Simpson's Rule :- This gives exact answer for all Polynomials of degree $n = 0, 1, 2, 3$.

- Error bounds indicate the class of polynomials for which these formulas are accurate.

Definition:- The degree of accuracy or precision of a quadrature formula is the largest positive integer n such that the formula is exact for $f(x) = x^k$ for each $k = 0, 1, \ldots n$.

- So Trapezoid / Midpoint rule : Precision one
  Simpson rule : Precision three

- Degree of precision is n if Error $= 0$ for all polynomials of degree $k = 0, 1, \ldots, n$ but is not zero for some polynomial of degree $n+1$.

Composite Quadrature or Numerical Integration

$f : [a,b] \to \mathbb{R}$ & $b-a$ is not small.

Divide $[a,b]$ into n-parts, n-even

let $x_i = a + \frac{b-a}{n} \cdot i$ $\quad i = 0, \ldots, n$ $\quad h = \frac{b-a}{n}$

Apply Rule / Quadrature :- to $[x_{2j-2}, x_{2j}]$

$j = 1, 2, \ldots, \frac{n}{2}$

Then sum the approximations to produce an approximation to $\int_a^b f(x)\,dx$.

$$I = \int_a^b f(x)\,dx = \sum_{j=1}^{n/2} \underbrace{\int_{x_{2j-2}}^{x_{2j}} f(t)\,dt}_{\text{apply rule to each term}}$$

Composite Midpoint Rule:

$$I = \sum_{j=1}^{n/2} \left[f\left(\frac{x_{2j}+x_{2j-2}}{2}\right) \cdot (2h) + \frac{(2h)^3}{24} f''(\xi_j) \right]$$

$$= 2h \sum_{j=1}^{n/2} f(x_{2j-1}) + \frac{8(b-a)^3}{24 n^3} \sum_{j=1}^{n/2} f''(\xi_j)$$

$$I = \underbrace{2\frac{(b-a)}{n} \sum_{j=1}^{n/2} f(x_{2j-1})}_{\text{Midpoint Rule}} + \underbrace{\frac{(b-a)^3}{6n^2} f''(\tilde{\xi})}_{\text{Error Composite midpoint}}$$

Composite Simpson Rule :-

$$I = \sum_{j=1}^{n/2} \left[\frac{h}{3} [f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})] - \frac{h^5}{2^5 \cdot 90} f^4(\xi_j) \right]$$

$$= \frac{h}{3} \left[f(a) + 2\sum_{j=2}^{n/2} f(x_{2j-2}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right]$$

$$- \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

$$= \frac{b-a}{3n} \underbrace{\left[f(a) + 2\sum_{j=2}^{n/2} f(x_{2j-2}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right]}_{\text{Composite Simpson rule}}$$

$$\underbrace{- \frac{(b-a)^5}{180} \cdot \frac{1}{n^4} f^{(4)}(\xi)}_{\text{Error Simpson Composite}}$$

Composite Trapezoid Rule :-   Here n - even or odd

$$I = \sum_{j=1}^{n} \left[\left[f(x_j) + f(x_{j-1}) \right] \frac{h}{2} - \frac{h^3}{12} f''(\xi_j) \right]$$

$$= \underbrace{\frac{b-a}{2n} \left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b) \right]}_{\text{Composite Trapezoid rule}} + \underbrace{\frac{(b-a)^3}{6n^2}}_{\text{Error Composite Trapezoid}} f''(\xi)$$